DIGITAL CONTROL

(Course Material)

4 AE-SE

INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE

LAAS CNRS

DIGITAL CONTROL

This course belongs to the "UF" *Data Acquisition Architecture Systems and Digital Control*

- **LECTURES:** Germain Garcia (10 sessions)
- **ASSOCIATED TUTORIALS:** Yassine Ariba (6 sessions)
- Evaluation : 1 written Exam, 1 practical work together with Data Acquisition Architecture Systems
- All the associated supports on Moodle Page

I4AEAU11_01 - Commande Numérique

- **PREREQUISITE**: Ordinary differential equations, Linear algebra, Basic course in signal theory, Analysis and control of linear continuous-time invariant systems.

Introduction Difference equation Z-Transform Transfer Function State-Space Model Some Model Transformations Summary Example

SOME REFERENCES

- *"Automatique Linéaire: Systèmes à Temps Discrets"* **B.Pradin, G. Garcia**, Course support of INSA , Toulouse, 2010-2011. *(Cover this course, available on the Moodle page)*
- *"Analog and Digital Control System Design"*. **C.T. Chen**, Saunders College Publishing, 2006. *(Very complete and covers many topics)*
- *"Discrete-Time Control Systems"*. **K. Ogata**. Prentice Hall. 1995, 2nd Edition. *(Covers many topics with a very pedagogical presentation)*
- *"Digital Control of Dynamic Systems"*. **G.F. Franklin, J. D. Powell, M.L. Workman**. Ellis-Kagle Press, 1997. *(Another interesting and complete book)*

DIGITAL CONTROL

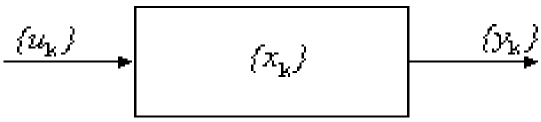
Chapter I - Discrete Models For Linear Time Invariant Systems

The objectives of this chapter are

- List the main properties of the linear invariant discrete models
- Present the three models: Difference (recurrent) equation, Transfer function, State-space model
- Relations between all the models

Table of Contents I

I.1 - Introduction



- The main assumptions are
  - Linearity
  - Invariance
  - Causality
- For the moment, we consider the independent variable $k \in \mathbb{Z}$ from a mathematical point of view. Later this variable will be the *discrete-time*.
- From the above assumptions, we can consider three different models: difference equation, transfer function and state-space model.

Table of Contents I

I.2 - Difference (Recurrent) Equation

From the adopted assumptions, a first model can be written as

$$a_n\, y_{k+n} + \cdots + a_1\, y_{k+1} + a_0\, y_k = b_m\, u_{k+m} + \cdots + b_1\, u_{k+1} + b_0\, u_k \qquad m \leqslant n$$

with initial conditions $y_0,\ y_1,\ \ldots,\ y_{n-1} \in \mathbb{R}$.

Its properties follow from assumptions.

- Linearity

$$\sum_{i=1}^{m} b_i \left(\sum_{j=1}^{2} \alpha_j u^j_{k+i} \right) = \sum_{j=1}^{2} \alpha_j \left(\sum_{i=1}^{m} b_i u^j_{k+i} \right) =$$
$$\sum_{j=1}^{2} \alpha_j \left(\sum_{i=1}^{n} a_i y^j_{k+i} \right) = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{2} \alpha_j y^j_{k+i} \right)$$

- Invariance because $a_i, b_j \in \mathbb{R}$ for $i = 1, \ldots, n,\ j = 1, \ldots m$
- Causality because $m \leqslant n$

I.1 - Difference (Recurrent) Equation

Introduce the shift operator q defined by

$$\begin{array}{rcl} qu_k & = & u_{k+1} \\ q^i u_k & = & u_{k+i} \end{array}$$

The difference equation can be written as

$$D(q)y_k = N(q)u_k$$

with

$$\begin{array}{rcl} D(q) & = & a_0 + a_1 q + \ldots + a_n q^n \\ N(q) & = & b_0 + b_1 q + \ldots + b_m q^m \end{array}$$

$D(q) = 0$ is *the characteristic equation* of the difference equation. Its roots are *the characteristic roots*.

Table of Contents I

I.3 - Z-Transform

We introduce a specific transform, the Z-Transform which is in some sense a transform similar as Laplace transform in the context of numerical series.

Definition

Consider $\{f_k\}_{k\in\mathbb{N}}$ a numerical serie. The Z-Transform of $\{f_k\}_{k\in\mathbb{N}}$ is the serie defined by

$$F(z) = \mathcal{Z}[\{f_k\}] = \sum_{k=0}^{+\infty} f_k\, z^{-k}, \quad z \in \mathbb{C}$$

Some conditions are needed for ensuring convergence of the series (convergence radius)

I.3 - Z-Transform

Its main properties are summarized below.

Properties of Z-Transform

1. Linearity

$$\mathcal{Z}[\alpha\{f_k\} + \beta\{g_k\}] = \alpha\,\mathcal{Z}[\{f_k\}] + \beta\,\mathcal{Z}[\{g_k\}]$$

2. Convolution product: The Z-Transform of the convolution product of two numerical series $\{f * g\}_k$ defined by

$$\sum_l f_l\, g_{n-l} = \sum_l f_{n-l}\, g_l$$

is given by

$$\mathcal{Z}[\{f * g\}_k] = F(z)\, G(z)$$

I.3 - Z-Transform

3. Shifting theorem

$$\mathcal{Z}[\{f_{k-l}\}] = z^{-l}\mathcal{Z}[\{f_k\}] = z^{-l}F(z)$$

4. Differentiation theorem

$$\mathcal{Z}[\{f_{k+l}\}] = z^{l}\left[\mathcal{Z}[\{f_k\}] - \sum_{i=0}^{l-1} f_i\, z^{-i}\right]$$

5. Initial value theorem

$$f_0 = \lim_{z\to\infty} F(z)$$

6. Final value theorem

$$\lim_{k\to\infty} f_k = \lim_{z\to 1}(1-z^{-1})\,F(z)$$

if the poles of $(1-z^{-1})F(z)$ are in the unit disk.

Table of Contents I

I.4 - Transfer Function

Taking the Z transform of the difference equation, we have

$$(a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n) Y(z) = (b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m) Y(z) + I(z)$$

where

$$\begin{aligned} I(z) &= 0 \text{ if the initial conditions are zero} \\ &\neq 0 \text{ if the initial conditions are non zero} \end{aligned}$$

and

$$Y(z) = \underbrace{\frac{N(z)}{D(z)}}_{\text{Transfer function}} U(z) + \frac{I(z)}{D(z)}$$

I.4 - Transfer Function

The transfer function can be written as

$$G(z) = \frac{N(z)}{D(z)} = \frac{b_m}{a_n} \frac{\prod_{i=1}^{m}(z - z_i)}{\prod_{i=1}^{n}(z - p_i)}$$

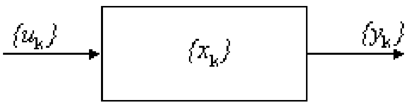
where

- The roots of numerator z_i are the zeros of the system
- The roots of dominator p_i are the zeros of the system
- n is the order of the system
- $D(z) = 0$ is the characteristic polynomial of the system

Table of Contents I

Introduction Difference equation Z-Transform Transfer Function State-Space Model Some Model Transformations Summary Example

I.5 - State-Space Model



The state space model is given by

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k, \quad x(0) = x_0 \end{cases}$$

where

- x_k is the state, u_k the input , y_k the output and x_0 the initial condition
- $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$
- A *is the dynamical matrix,* B *the input matrix,* C *the output matrix and* D *the feedforward matrix*
- We consider *a single-input single-output (SISO) system, i.e.* $m = p = 1$

I.5 - State-Space Model: Non-uniqueness of state-space model

Consider

$$\left\{\begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k + Du_k, \quad x(0) = x_0 \end{array}\right.$$

and consider a new state vector $x_k = M\bar{x}_k$. Then

$$M\bar{x}_{k+1} = AM\bar{x}_k + Bu_k \Rightarrow \boxed{\left\{\begin{array}{rcl} \bar{x}_{k+1} & = & M^{-1}AM\bar{x}_k + M^{-1}Bu_k \\ y_k & = & CM\bar{x}_k + Du_k, \quad \bar{x}(0) = M^{-1}x_0 \end{array}\right.}$$

Selecting appropriately M, we can obtain particular state-space models

- Diagonal form
- Canonical form (controllability and observability)

Table of Contents I

I.6 - Some Model Transformations: From TF to SS

Consider a transfer function

$$G(z) = \frac{b_0 + b_1 z + \cdots + b_m z^m}{a_0 + a_1 z + \cdots + a_n z^n} = \frac{N(z)}{D(z)}, \quad m < n$$

If the poles are distinct

$$G(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{a_n(z-p_1)(z-p_2)\cdots(z-p_n)} = \sum_{i=1}^{n} \frac{\alpha_i}{z-p_i}$$

The matrices of a state-space model are

$$\begin{cases} A = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} & B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \\ C = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix} & \alpha_i = \beta_i \gamma_i \end{cases}$$

I.6 - Some Model Transformations: From TF to SS

If the poles are multiple

$$G(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(z-\lambda)^n} = \frac{\alpha_1}{z-\lambda} + \frac{\alpha_2}{(z-\lambda)^2} + \cdots + \frac{\alpha_n}{(z-\lambda)^n}$$

The matrices of a state-space model are

$$A = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \alpha_n & \alpha_{n-1} & \cdots & \alpha_1 \end{bmatrix}$$

I.6 - Some Model Transformations: From TF to SS

If $a_n = 1$,the two canonical forms can be obtained

Controllability canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \ddots & \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & \ldots & b_m & 0 & \cdots \end{bmatrix}$$

Observability canonical form

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & & \ddots & \ddots & \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

I.6 - Some Model Transformations: From TF to SS

When $m = n$, the transfer function is given by

$$G(z) = \frac{N(z)}{D(z)} = \frac{b_0 + \cdots + b_n z^n}{a_0 + \cdots + a_{n-1} z^{n-1} + z^n}$$

We can write

$$N(z) = N^*(z) + d\,D(z) \text{ with } d = b_n \text{ and } N^*(z) = N(z) - b_n D(z)$$

And

$$G(z) = \frac{N(z)}{D(z)} = \frac{N^*(z)}{D(z)} + d$$

Then

$$\boxed{G^*(s) \rightarrow (A, B, C) \qquad G(s) \rightarrow (A, B, C, b_n)}$$

I.6 - Some Model Transformations: From SS to TF

$$\begin{aligned} zX(z) - zx_0 &= AX(z) + BU(z) \\ (zI - A)X(z) &= BU(z) + zx_0 \\ X(z) &= (zI - A)^{-1}BU(z) + (zI - A)^{-1}zx_0 \end{aligned}$$

$$\begin{aligned} Y(z) &= CX(z) + DU(z) \\ &= \left[C(zI - A)^{-1}B + D\right] U(z) + \underbrace{C(zI - A)^{-1}zx_0}_{0 \text{ if C.I are zero}} \\ &= G(z)U(z) + \underbrace{\frac{I(z)}{D(z)}}_{0 \text{ if C.I are zero}} \end{aligned}$$

Then

$$\boxed{G(z) = C(zI - A)^{-1}B + D}$$

I.6 - Some Model Transformations: From SS to SS

The characteristic polynomial is

$$\begin{aligned} P(z) &= \det(zI_n - A) \\ &= z^n + a_{n-1}\, z^{n-1} + \cdots + a_1\, z + a_0 \end{aligned}$$

The matrix M transforming the original model into *the controllability canonical form* is

$$M = [m_1 \quad \cdots \quad m_n]$$

with

$$\begin{aligned} m_n &= B \\ m_{n-1} &= (A + a_{n-1}\, I_n)\, B \\ m_{n-2} &= (A^2 + a_{n-1}\, A + a_{n-2}\, I_n)\, B \\ &\cdots \\ m_1 &= (A^{n-1} + a_{n-1}\, A^{n-2} \cdots + a_1\, I_n)\, B \end{aligned}$$

(Show it. Hint: $AM = MA_c$ and $MB_c = B$ with $M = [m_1 \quad \cdots \quad m_n]$)

I.6 - Some Model Transformations: From SS to SS

The matrix M_o transforming the original model into *the observability canonical form* is

$$M_o = \left([m_1 \quad \cdots \quad m_n]'\right)^{-1}$$

with

$$\begin{aligned} m_1 &= C' \\ m_2 &= (A' + a_{n-1}\, I_n)\, C' \\ m_3 &= ((A')^2 + a_{n-1}\, A' + a_{n-2}\, I_n)\, C' \\ &\cdots \\ m_n &= ((A')^{n-1} + a_{n-1}\,(A')^{n-2} \cdots + a_1\, I_n)\, C' \end{aligned}$$

(Show it. Hint: $A'(M_o')^{-1} = (M_o')^{-1} A_o'$ *and* $(M_o')^{-1} C_o' = C'$ *with* $(M_o')^{-1} = [m_1 \quad \cdots \quad m_n]$*)*

Table of Contents I

I.7 - Summary

Difference Equation

$$a_n y_{k+n} + \cdots + a_1 y_{k+1} + a_0 y_k = b_m u_{k+m} + \cdots + b_1 u_{k+1} + b_0 u_k$$

$$D(z) y_k = N(z) u_k$$

$$m \leqslant n$$

Characteristic Equation:

$$a_n r^n + \cdots + a_1 r + a_0 = 0$$

System Order: n

Poles $\equiv$ **Roots of Characteristic Equation**:

$$r_i \quad i = 1, \ldots, n$$

Transfer Function

$$G(z) = \frac{N(z)}{D(z)}$$

$$G(z) = \frac{b_m z^m + \cdots + b_1 z + b_0}{a_n z^n + \cdots + a_1 z + a_0}$$

$$m \leqslant n$$

Characteristic Polynomial:

$$a_n z^n + \cdots + a_1 z + a_0 = 0$$

System Order: n

Poles $\equiv$ **Roots of Characteristic Polynomial**

$$G(z) = \frac{b_m}{a_n} \frac{\prod_{j=1}^{m}(z - z_j)}{\prod_{i=1}^{n}(z - p_i)}$$

State-Space Model

$$\begin{cases} x_{k+1} = A x_k + B u_k \\ y_k = C x_k + D u_k \end{cases}$$

Characteristic Polynomial of A:

$$P(\lambda) = \mathbf{det}(\lambda I - A)$$

System Order: $n = \mathbf{dim}(A)$

Poles $\equiv$ **Eigenvalues of** A $\equiv$ **roots of Characteristic Polynomial**;

$$\lambda_i \quad i = 1, \ldots, n$$

Table of Contents I

I.8 - Example

Consider the system described by the difference equation

$$y_{k+2} + 3y_{k+1} + 2y_k = u_k, \quad y_0 = 1, \; y_1 = 1$$

Taking the Z-transform, we have

$$z^2Y(z) - z^2y_0 - zy_1 + 3zY(z) - 3zy_0 + 2Y(z) = U(z)$$

Replacing by the values of $y_0$and y_1

$$(z^2 + 3z + 2)Y(z) = U(z) + z^2 + 4z$$

and

$$Y(z) = \underbrace{\frac{1}{(z+1)(z+2)}}_{G(z)} U(z) + \frac{z^2 + 4z}{z+1)(z+2)}$$

The order is 2. The poles are: -1 and -2. There is no finite zero.

Introduction Difference equation Z-Transform Transfer Function State-Space Model Some Model Transformations Summary Example

I.8 - Example

Taking as state variables $x_{1_k} = y_k$ and $x_{2_k} = y_{k+1}$, we obtain

$$\begin{cases} x_{k+1} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \end{cases}$$

The transfer function is recovered as

$$\begin{aligned} G(z) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{z^2 + 3z + 2} \\ &= \frac{1}{(z+1)(z+2)} \end{aligned}$$

I.8 - Example

The eigenvalues are -1 and -2. With

$$M = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

The transformed system is :

$$\begin{cases} x_{k+1} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k \end{cases}$$

The diagonal form can be obtained from the transfer function

$$G(z) = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} + \frac{-1}{z+2}$$

Then

$$\begin{cases} x_{k+1} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 1 & -1 \end{bmatrix} x_k \end{cases}$$

I.8 - Example

The canonical forms can be deduced from the transfer function.

Controllability canonical form

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \end{cases}$$

Observability canonical form

$$\begin{cases} \bar{x}_{k+1} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \bar{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}_k \end{cases}$$

DIGITAL CONTROL

Chapter II - Models of Sampled-Data Systems

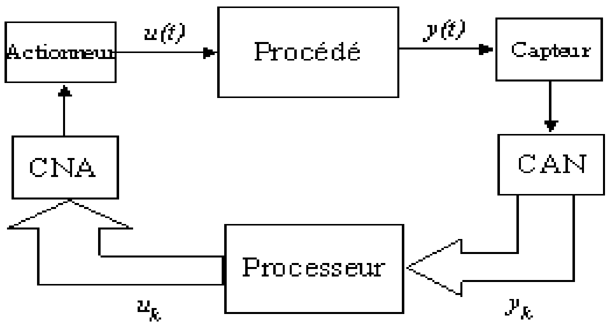
The objectives of this chapter are

- How to obtain a sampled-data model for a continuous-time system
- The available models of the continuous-time system are the transfer function or a state-space model
- The deduced sampled-data models are the transfer function or a state-space model

Table of Contents I

II.1 - Introduction

A digital control system is represented in the following figure



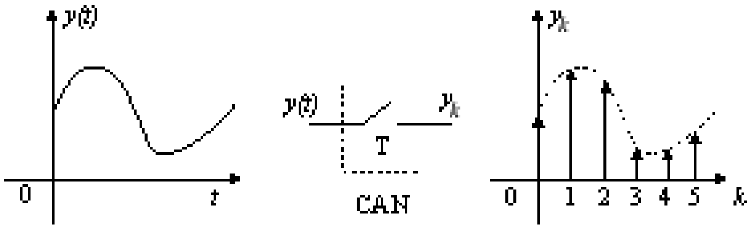
- To communicate with the real world, A/D and D/A conversions are needed.

$$A/D : y(t) \rightarrow y(kT) \text{ and } D/A : u(kT) \rightarrow u(t)$$

- A sampling frequency has to be selected.
  Shannon Theorem: *If the signals spectra contain frequencies in the frequency band* $[0, \omega_B]$, *the sampling frequency has to be chosen at least equal to* $2\omega_B$.
- In practice and due to the feedback, the sampling frequency is chosen 5 to 10 times the maximal frequency contained in the system bandwidth.
- To prevent *aliasing*, *anti-aliasing filters* are also needed

Table of Contents I

II.2 - A/D Conversion



- The A/D conversion can be represented as the analog signal modulated by a pulse train

$$y^*(t) = y(t)\,\delta_T(t), \quad \delta_T(t) = \sum_{k=0}^{+\infty} \delta(t-kT)$$

$$y^*(t) = \sum_{k=0}^{+\infty} y(t)\,\delta(t-kT) = \sum_{k=0}^{+\infty} y_k\,\delta(t-kT)$$

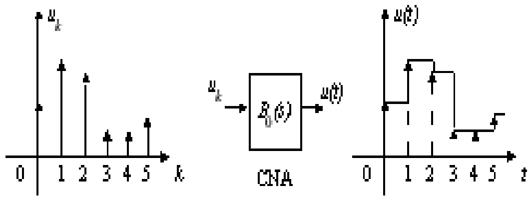
$$y_k = y(kT) \text{ : sample of } y(t) \text{ à } t{=}kT$$

- The sampled signal is in fact the sequence $y(kT)$

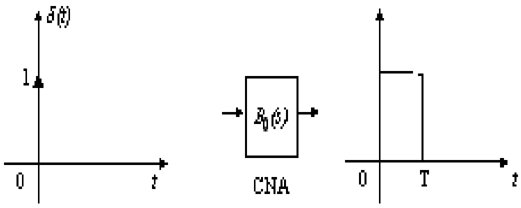
$$\boxed{\{y(kT)\} \equiv \{y(k)\} \equiv \{y_k\}}$$

Table of Contents I

II.3 - D/A Conversion



- From the values u_k, the A/D conversion produces a continuous-time signal constant on a period T *(Zero-order hold)*.
- To determine the transfer function associated with the zero-order hold, remark that

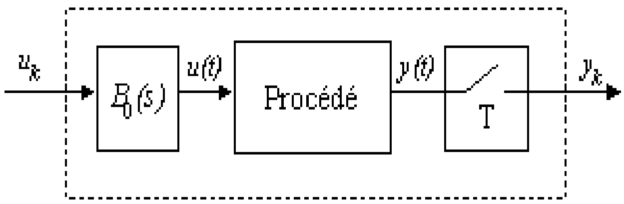


– The transfer function $B_0(s)$ is the Laplace transform of the previous impulse response.

$$B_0(s) = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s}$$

Table of Contents I

II.4 - Transfer Function of a Sampled-Data System



- Transfer function of continuous-time system $\overset{?}{\rightarrow}$ Z-transfer function of sampled-data system.

$$G_c(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]} \quad \overset{?}{\rightarrow} \quad G(z) = \frac{\mathcal{Z}[y_k]}{\mathcal{Z}[u_k]}$$

- In the continuous-time domain, the sequence $\{u_k\}$, can be represented as

$$u^*(t) = \sum_{k=0}^{+\infty} u_k\, \delta(t - kT)$$

II.4 - Transfer Function of a Sampled-Data System

We have

$$U(s) = B_0(s)\,U^*(s) \text{ with } U^*(s) = \mathcal{L}[u^*(t)] = \sum_{k=0}^{+\infty} u_k\, e^{-kTs}$$

Then

$$Y(s) = G_c(s)\,U(s) = \underbrace{G_c(s)\,B_0(s)}_{G_{bc}(s)}\,U^*(s) \text{ with } Y(s) = \sum_{k=0}^{+\infty} G_{bc}(s)\,u_k\, e^{-kTs}$$

We can also write

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \sum_{k=0}^{+\infty} g_{bc}(t-kT)\,u(kT)$$

with $g_{bc}(t)$ inverse Laplace transform of $G_{bc}(s)$.

II.4 - Transfer Function of a Sampled-Data System

Taking the samples a times nT of the previous signal, we have

$$y(nT) = \sum_{k=0}^{+\infty} g_{bc}[(n-k)T]\,u(kT)$$

which is nothing but a discrete convolution product. Then by the property of Z-transform

$$Y(z) = G(z)\,U(z) \text{ with } G(z) = \mathcal{Z}\{G_{bc}(s)\} = \mathcal{Z}\{B_0(s)\,G_c(s)\}$$

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}\,G_c(s)\right\}$$

By the linearity of the Z-transform

$$\boxed{G(z) = (1-z^{-1})\,\mathcal{Z}\left\{\frac{G_c(s)}{s}\right\} = \frac{z-1}{z}\,\mathcal{Z}\left\{\frac{G_c(s)}{s}\right\}}$$

II.4 - Transfer Function of a Sampled-Data System

Another important implication can be deduced. Consider a signal $s(t)$ and its sampled version

$$s^*(t) = \sum_{k=0}^{+\infty} s_k \, \delta(t - kT)$$

The Laplace transform of $s^*(t)$ is given by

$$\mathcal{L}[s^*(t)] = \sum_{k=0}^{+\infty} s_k \, e^{-kTs}$$

Also remark that $e^{Ts}s(t) = s(t+T)$. If $t = kT$, we have $e^{Ts}s(kT) = s(kT+T)$ which can also be written $e^{Ts}s_k = s_{k+1}$. And we conclude that

$$S(z) = \mathcal{L}[s^*(t)] = \sum_{k=0}^{+\infty} s_k \, \underbrace{e^{-kTs}}_{z^{-k}} \text{ and then } z = e^{Ts}$$

II.4 - Transfer Function of a Sampled-Data System

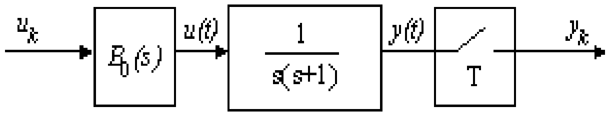
- The complex function e^{Ts} is a *multiform function* leading to the *aliasing phenomenon*
- From a frequency point of view, the aliasing results from the relation $z = e^{Tj\omega}$.
- Because of periodicity of the complex function $e^{Tj\omega}$, we see that the sampling process does not discriminate the frequencies ω in the bands

$$[-k\pi/T, k\pi/T], \; k \in \mathbb{N}$$

  generating the aliasing phenomenon.
- This means that the maximal frequency for a sampled signal is π/T. This frequency is called *the Nyquist frequency* and it corresponds to $z = e^{\pi} = -1$.
- The Nyquist frequency is half of the sampling frequency.

II.4 - Transfer Function of a Sampled-Data System

EXAMPLE



$$G(z) = \mathcal{Z}[B_0(s)\, G_c(s)] = \frac{z-1}{z}\, \mathcal{Z}\left[\frac{G_c(s)}{s}\right]$$

$$\frac{G_c(s)}{s} = \frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

Using the Z-transform table, we have

$$G(z) = \frac{z-1}{z}\left[-\frac{z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}\right]$$

II.4 - Transfer Function of a Sampled-Data System

EXAMPLE: (Continued)

and

$$\boxed{G(z) = \frac{K(z-b)}{(z-1)(z-a)}}$$

with

$$K = e^{-T} - 1 + T$$
$$a = e^{-T}$$
$$b = 1 - \frac{T\,(1-e^{-T})}{e^{-T} - 1 + T}$$

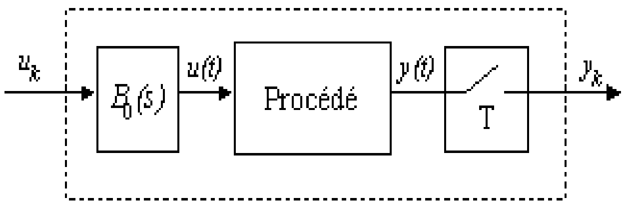
Numerical Application:

If $T = 1s.$ then

$$G(z) = 0,3679\, \frac{z + 0,7183}{(z-1)(z-0,3679)}$$

Table of Contents I

II.5 - State-Space Model of a Sampled-Data System



- State-space model of continuous-time system $\overset{?}{\rightarrow}$ State-space model of sampled-data system

$$(A_c, B_c, C_c, D_c) \quad \overset{?}{\rightarrow} \quad (A, B, C, D)$$

- The state-space model of the system is

$$\begin{cases} \dot{x}(t) &= A_c\, x(t) + B_c\, u(t) \\ y(t) &= C_c\, x(t) + D_c\, u(t) \end{cases}$$

II.5 - State-Space Model of a Sampled-Data System

We have

$$x(t) = e^{A_c(t-t_0)}\,x_0 + \int_{t_0}^{t} e^{A_c(t-\tau)}\,B_c\,u(\tau)\,d\tau$$

On the interval $[kT, (k+1)T]$, i.e. $t_0 = kT$ and $t = (k+1)T$ and from the zero-order hold

$$x_{k+1} = e^{A_c T}\,x_k + \int_{kT}^{(k+1)T} e^{A_c((k+1)T-\tau)}\,B_c\,u_k\,d\tau$$

By the change of variable $\alpha = (k+1)T - \tau$

$$x_{k+1} = e^{A_c T}\,x_k + \left\{\int_0^T e^{A_c\alpha}\,d\alpha\right\} B_c\,u_k$$

The output of the sampled-data system is $y(kT)$, then $C = C_c$ and $D = D_c$, and

$$\left\{\begin{array}{lcccc} x_{k+1} & = & A\,x_k & + & B\,u_k \\ y_k & = & C\,x_k & + & D\,u_k \end{array}\right.$$

$$A = e^{A_c T},\; B = \int_0^T e^{A_c\alpha} B_c\,d\alpha,\; C = C_c,\; D = D_c$$

Introduction A/D Conversion D/A Conversion Z-Transfer Function of a Sampled-Data System State-Space Model of a Sampled-Data System

II.5 - State-Space Model of a Sampled-Data System

- We have

$$\text{Eigenvalues of } A_c : \lambda_i,\; i = 1, \cdots, n$$
$$\downarrow$$
$$\text{Eigenvalues of } A = e^{A_c T} : e^{\lambda_i T},\; i = 1, \cdots, n$$

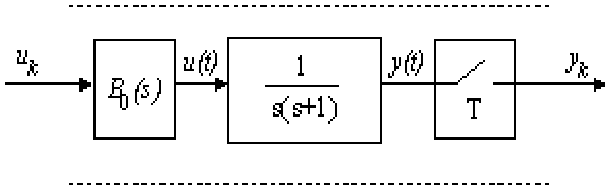
The sampling process does not change the open-loop stability property.

- If A_c invertible (Show it)

$$B = A_c^{-1}\,(e^{A_c T} - I)\,B_c = (e^{A_c T} - I)A_c^{-1}\,B_c$$

II.5 - State-Space Model of a Sampled-Data System

EXAMPLE



A state space-model for the system is (Show it)

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(t) \end{cases}$$

For $T = 1s$, we have

$$A = e^{A_c} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-1} \end{bmatrix}$$

$$B = \int_0^1 e^{A_c \alpha} B_c \, d\alpha = \int_0^1 \begin{bmatrix} 1 \\ e^{-\alpha} \end{bmatrix} d\alpha = \begin{bmatrix} 1 \\ 1 - e^{-1} \end{bmatrix}$$

II.5 - State-Space Model of a Sampled-Data System

EXAMPLE (Continued)

The sampled-data state-space model is given by

$$\begin{cases} x_{k+1} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-1} \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 - e^{-1} \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 1 & -1 \end{bmatrix} x_k \end{cases}$$

If we compute the transfer function, we recover the result of the previous section

$$G(z) = C \left(zI_n - A \right)^{-1} B$$

$$G(z) = 0,3679 \frac{z + 0,7183}{(z-1)(z-0,3679)}$$

DIGITAL CONTROL

Chapter III - Response of discrete-time linear system

The objectives of this chapter are

- Compute the response of the discrete-time linear system
- From all the models presented in the previous chapters
- In the case of sampled-data systems, discuss the relations with the original continuous-time system

Table of Contents I

III.1 - Introduction

- Evaluate the responses for a given system is important to analyze the system behavior

- Among the input signals of interest, some of them are particularly important: impulse, step, ramp, periodic signals...

- In that context, it is possible, for the considered models, to derive the responses analytically. But in practical situations, they are obtained numerically

- However, the principles of analytical determinations are important because they allow to identify the key parameters whose values have an impact on the responses

Table of Contents I

III.2 - From the difference equation

- The difference equation recalled here

$$a_n\, y_{k+n} + \cdots + a_1\, y_{k+1} + a_0\, y_k = b_m\, u_{k+m} + \cdots + b_1\, u_{k+1} + b_0\, u_k, \quad m \leqslant n$$

with initial conditions $y_0,\ y_1,\ \ldots, y_{n-1}$ can be interpreted as an algorithm directly adapted for a numerical simulation.

- A method very similar to the one existing for linear time-invariant differential equations exists in the context of linear time-invariant difference equations, but more involved. it can be used for analytical calculations.

III.2 - From the difference equation

EXAMPLE

$$y_{k+2} - 3\,y_{k+1} + 2\,y_k = u_k$$

with

$$y_k = 0 \quad \forall k \leqslant 0 \quad \text{et} \quad u_k = 0 \quad \forall k \neq 0 \quad \text{et} \quad u_0 = 1$$

$$u_k = 0 \quad \forall k \neq 0 \quad \text{et} \quad u_0 = 1$$

Iterating from the initial conditions

$$\begin{aligned} y_2 &= 3y_1 - 2y_0 + u_0 &= 1 \\ y_3 &= 3y_2 - 2y_1 + u_1 &= 3 \\ y_4 &= 3y_3 - 2y_2 + u_2 &= 7 \end{aligned}$$

Here we can deduce a closed-form expression (in general this is not obvious)

$$y_k = -1 + 2^{k-1} \quad \forall k > 1$$

Table of Contents I

III.3 - From The Transfer Function $G(z)$

Consider the transfer function

$$\frac{Y(z)}{U(z)} = \frac{N(z)}{D(z)} = G(z)$$

If the case of non zero initial conditions, the initial conditions must be included. Then

$$\frac{Y(z)}{U(z)} = \frac{N(z)}{D(z)} \rightarrow D(z)Y(z) = N(z)U(z) \xrightarrow{\mathcal{Z}^{-1}(.)}$$

$$\text{Difference equation} \xrightarrow{\mathcal{Z}(.) \text{ with C.I}} Y(z) \text{ is a rational function } \rightarrow$$

$$\text{Decomposition of } Y(z) \xrightarrow{\mathcal{Z}^{-1}(.) \text{ using table of Z-transform}} y_k$$

III.3 - From The Transfer Function $G(z)$

EXAMPLE (Continued)

The initial conditions are zero then

$$Y(z) = G(z)U(z) = \frac{U(z)}{z^2 - 3z + 2}$$

with $U(z) = \sum_{i=0}^{\infty} u_k z^{-k} = 1$. We have

$$Y(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{z-2} - \frac{1}{z-1}$$

and

$$Y(z) = z^{-1} \underbrace{\left[\frac{z}{z-2} - \frac{z}{z-1}\right]}_{Z^{-1}[.] \rightarrow 2^k - 1^k}$$

Then

$$y_k = 2^{k-1} - 1^{k-1} = 2^{k-1} - 1$$

Table of Contents I

III.4 - From The State-Space Model

It is possible to use the following formula

$$Y(z) = C(zI - A)^{-1}x_0 + \left[C(zI - A)^{-1}B + D\right]U(z)$$
$$Y(z) = C(zI - A)^{-1}x_0 + G(z)U(z)$$

and work with the Z-transform. From the state-space model

$$\begin{cases} x_{k+1} = A\,x_k + B\,u_k \\ y_k = C\,x_k + D\,u_k, \quad x_0 \end{cases}$$

Iterating from a state x_m, we have

$$\begin{aligned} x_{m+1} &= A\,x_m + B\,u_m \\ x_{m+2} &= A\,x_{m+1} + B\,u_{m+1} \\ &= A^2\,x_m + A\,B\,u_m + B\,u_{m+1} \\ &\dots \\ x_k &= A^{k-m}\,x_m + \sum_{j=m}^{k-1} A^{k-1-j}\,B\,u_j \end{aligned}$$

III.4 - From The State-Space Model

If $x_m = x_0$, we obtain

$$\boxed{x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j}$$

and

$$\boxed{y_k = CA^k x_0 + C \sum_{j=0}^{k-1} A^{k-1-j} B u_j + D u_k}$$

III.4 - From The State-Space Model: Computation of A^k

- $A = M\Lambda M^{-1} \rightarrow A^k = M\Lambda^k M^{-1} = \sum_{i=1}^{n} v_i w_i^T \lambda_i$ where

$$M = [v_1 \cdots v_n] \qquad M^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} \Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

- Another way for computing A^k

$$z X(z) - z x_0 = A X(z)$$

$$X(z) = (zI - A)^{-1} z x_0$$

Taking the inverse of the Z-transform of the previous expression, we obtain

$$x_k = A^k x_0$$

and

$$\boxed{A^k = \mathcal{Z}^{-1}\left[(zI - A)^{-1} z\right]}$$

III.4 - From The State-Space Model

EXAMPLE (Continued)

Taking a state-space vector $x_k = [y_k, y_{k+1}]^T$, a state-space model is given by

$$\left\{ \begin{array}{lcl} x_{k+1} & = & \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ y_k & = & \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \end{array} \right. , \quad x_0 = 0$$

The matrix M diagonalizing the dynamical matrix is given by

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then

$$y_k = C\,A^{k-1}\,B\,u_0 = CM\Lambda^{k-1}M^{-1}B\,u_0 = [1 \quad 0] \begin{bmatrix} 2-2^{k-1} & -1+2^{k-1} \\ 2-2^k & -1+2^k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{y_k = 2^{k-1} - 1}$$

DIGITAL CONTROL

Chapter IV - Stability

The objectives of this chapter are

- Identify the equilibrium states of a discrete-time system
- Give the conditions of stability of an equilibrium state
- Present algebraic criteria for stability
- The use of root locus in that context
- Introduce the notion of mode

Table of Contents I

IV.1 - Equilibrium States

Consider the system described in the state-space

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{cases}$$

Definition

A state $\bar{x}$ is *an equilibrium state* or *an equilibrium point* if when $u_k = 0$, the system being on $\bar{x}$, it remains on $\bar{x}$ indefinitely.

An equilibrium state $\bar{x}$ is a fixed point, i.e

$$x_{k+1} = x_k = \bar{x} \to A\bar{x} = \bar{x}$$

and

$$\boxed{(A - I)\bar{x} = 0}$$

IV.1 - Equilibrium States

- If matrix $A - I$ is regular, i.e.

$$\boxed{\det(A - I) \neq 0}$$

The unique equilibrium point is the origin $\bar{x} = 0$.

- If matrix $A - I$ is singular i.e.

$$\boxed{\det(A - I) = 0}$$

The number of equilibrium points is infinite

IV.1 - Equilibrium States

Examples

$$x_{k+1} = A\,x_k = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} x_k$$

$$\det(A - I) = -1$$

One one equilibrium point, the origin.

Exemple 2

$$x_{k+1} = A\,x_k = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x_k$$

$$\det(A - I) = 0 \longrightarrow \text{ Etats d'équilibre :}$$

$$A\,\bar{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \bar{x} = \bar{x} \longrightarrow \bar{x} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

Table of Contents I

IV.2 - Stability of Equilibrium States

Definition

The equilibrium point $\bar{x}$ is said *stable in the sense of Lyapunov* or (*stable*), if for all $\epsilon > 0$, there exists $r > 0$ which can depend on ϵ but independent of k such that

$$\|x_0 - \bar{x}\| < r \Rightarrow \|x_k - \bar{x}\| < \epsilon \quad \forall\, k > 0$$

Otherwise the equilibrium point will be said *unstable*

Th previous notion of stability can be insufficient in some practical situations. We introduce a strengthened notion of stability called *asymptotic stability*

Definition

The equilibrium point $\bar{x}$ is said *asymptotically stable* if

- $\bar{x}$ is stable
- There exists $r > 0$ such that $\|x_0 - \bar{x}\| < r \Rightarrow \lim_{k\to\infty} x_k \to \bar{x}$

IV.2 - Stability of Equilibrium States

$$x_{k+1} = A\,x_k \qquad x_0 \neq 0 \quad \longrightarrow \quad x_k = A^k x_0$$

We have

$$x_k = A^k x_0 = M\Lambda^k M^{-1} x_0$$

- For distinct eigenvalues

$$M = [v_1 \cdots v_n] \qquad M^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} e^{\Lambda t} = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

$$x_k = \sum_{i=1}^{n} v_i \lambda_i^k w_i^T x_0 = \sum_{i=1}^{n} N_i x_0 \lambda_i^k$$

IV.2 - Stability of Equilibrium States

- A possesses r distinct eigenvalues λ_i and s Jordan blocks $L_i i$ and $x_k = A^k x_0 = MJ^k M^{-1} x_0$

$$J = \begin{bmatrix} L_1 & & \\ & \ddots & \\ & & L_s \end{bmatrix} \qquad L_j = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}$$

$$J^k = \begin{bmatrix} L_1^k & & \\ & \ddots & \\ & & L_s^k \end{bmatrix} \quad L_j^k = \begin{bmatrix} \lambda_j^k & \frac{k\lambda_j^{k-1}}{1!} & \frac{k(k-1)\lambda_j^{k-2}}{2!} & \cdots \\ & \lambda_j^k & \frac{k\lambda_j^{k-1}}{1!} & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

$$x_k = \sum_{i=1}^{r} \sum_{j=1}^{k} N_i(j) x_0 \lambda_i^k$$

IV.2 - Stability of Equilibrium States

$$x_{k+1} = A\, x_k$$

A : $n \times n$ matrix, whose eigenvalues are $\lambda_1, \cdots, \lambda_r$.

1. Si $\exists j \in \{1, \cdots, r\}$ such that $|\lambda_j| > 1$, then $\bar{x} = 0$ est instable.
2. If $\forall j = 1, \cdots, r$, $|\lambda_j| \leqslant 1$, then
   - (a) If $\forall j = 1, \cdots, r$, $|\lambda_j| < 1$, then $\bar{x} = 0$ is asymptotically stable,
   - (b) if $\exists j \in \{1, \cdots, r\}$ such that $|\lambda_j| = 1$ and the multiplicity order of λ_j equal 1, then $\bar{x} = 0$ is stable,
   - (c) if $\exists j \in \{1, \cdots, r\}$ such that $|\lambda_j| = 1$ and the multiplicity order of λ_j is greater than 1,
     - (α) If the Jordan blocks associated with λ_j are scalar, then $\bar{x} = 0$ is stable,
     - (β) If $\exists$ non scalar Jordan blocks associated with λ_j , then $\bar{x} = 0$ is unstable.

IV.2 - Stability of Equilibrium States

Example

$$A = \begin{bmatrix} 0,5 & 0 \\ 0 & 0,25 \end{bmatrix}$$

$|\lambda_i| < 1.$ System asymptotically stable.

Example

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$|\lambda_i| = 1$ distinct. Stable.

Example

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = 2 > 1.$ System unstable.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\lambda = 1$ double . System unstable.

Table of Contents I

IV.3 - Algebraic Criteria: JURY Criterion

Consider a polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

The *JURY criterion* gives a necessary and sufficient conditions for the modulus of the roots of $P(z)$ to be strictly lower than unity. Its general formulation is complex. Here are the conditions for orders 2, 3 and 4.

$$n = 2 \quad \begin{cases} a_0 + a_1 + a_2 & > 0 \\ a_0 - a_1 + a_2 & > 0 \\ a_2 - a_0 & > 0 \end{cases} \qquad n = 3 \quad \begin{cases} a_0 + a_1 + a_2 + a_3 & > 0 \\ -a_0 + a_1 - a_2 + a_3 & > 0 \\ a_3 - |a_0| & > 0 \\ a_0 a_2 - a_1 a_3 - a_0^2 + a_3^2 & > 0 \end{cases}$$

$$n = 4 \quad \begin{cases} a_0 + a_1 + a_2 + a_3 + a_4 > 0 \\ a_0 - a_1 + a_2 - a_3 + a_4 > 0 \\ a_4^2 - a_0^2 - |a_0 a_3 - a_1 a_4| > 0 \\ (a_0 - a_4)^2 (a_0 - a_2 + a_4) + (a_1 - a_3)(a_0 a_3 - a_1 a_4) > 0 \end{cases}$$

IV.3 - Algebraic Criteria: JURY Criterion

EXAMPLE

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ -K & 0 & 0,25 \\ 1 & 3 & 0 \end{bmatrix} x_k$$

$$P(z) = \det(zI - A) = z^3 + (K - 0,75)\, z - 0,25$$

JURY Criterion

$$\begin{cases} a_0 + a_1 + a_2 + a_3 & = & K & > 0 \\ -a_0 + a_1 - a_2 + a_3 & = & K + 0,5 & > 0 \\ a_3 - |a_0| & = & 1 - 0,25 & > 0 \\ a_0 a_2 - a_1 a_3 - a_0^2 + a_3^2 & = & -K + 1,6875 & > 0 \end{cases}$$

Then $0 < K < 1,6875$

IV.3 - Algebraic Criteria: ROUTH Criterion

Introduce the bilinear transformation

$$z = \frac{1+w}{1-w} \implies w = \frac{z-1}{z+1}$$

If $z = \alpha + j\beta$, then

$$w = \frac{\alpha^2 + \beta^2 - 1 + 2j\beta}{(\alpha+1)^2 + \beta^2}$$

We conclude that

$$|z| < 1 \iff \alpha^2 + \beta^2 < 1 \iff \Re(w) < 0$$

$$P(z) \xrightarrow{\text{Bilinear Transformation}} Q(w)$$

$$|\text{ Roots } P(z)| < 1 \Rightarrow \Re(\text{Roots } Q(w)) < 0$$

It is possible to check the sign of the real part of roots of $Q(w)$ by ROUTH criterion.

IV.3 - Algebraic Criteria: ROUTH Criterion

EXAMPLE (Continued)

$$P(z) = \det(zI - A) = z^3 + (K - 0,75)\,z - 0,25$$

$$Q(w) = w^3(K + 0,5) + w^2(3 - K) + w(4,5 - K) + K$$

ROUTH table :

$$\begin{array}{lcc} w^3 & K+0,5 & 4,5-K \\ w^2 & 3-K & K \\ w^1 & \dfrac{-8K+13,5}{3-K} & \\ w^0 & K & \end{array}$$

Then by ROUTH criterion

$$\begin{cases} K+0,5 & >0 \\ 3-K & >0 \\ 4,5-K & >0 \\ K & >0 \\ -8K+13,5 & >0 \end{cases}$$

Then, $0 < K < 1,6875$

Table of Contents I

IV.4 - Root Locus

$$P(z) = z^3 + (K - 0,75)\,z - 0,25$$

$$1 + \frac{Kz}{(z-1)(z+0.5)} = 0$$

- The root locus can be used as for continuous-time systems
- Only the region of stability and the relations between poles values and associated dynamics change
- The simple rules to deduce approximately the root locus are summarized in classical courses on Control System Design (see for example the Pradin's et al. Book).
- The root locus can be obtained by efficient numerical tools (MATLAB for example)

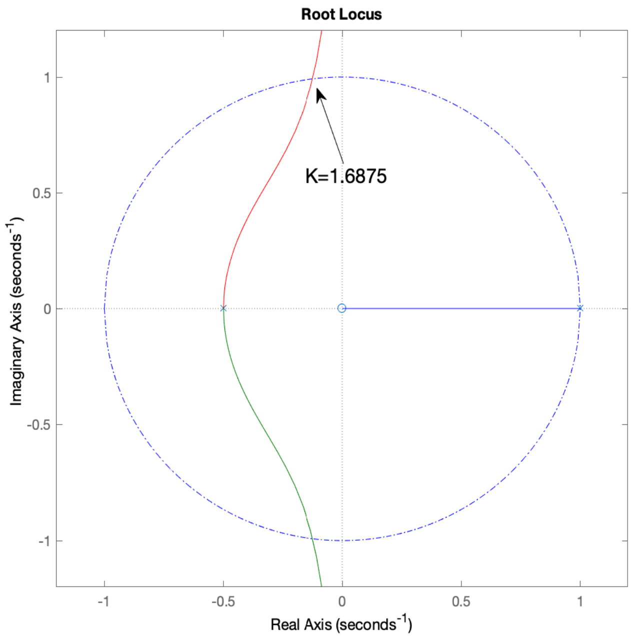


Table of Contents I

IV.5 - Stability of Sampled-Data Systems: Open-Loop

Continuous-time Plant:

$$\begin{cases} \dot{x}(t) = A_c\, x(t) + B_c\, u(t) \\ y(t) = C_c\, x(t) \end{cases}$$

N. S. Cond. Of Asymptotic Stab. :

$$\Re(\lambda_i) < 0 \qquad \forall\, \lambda_i \in \text{Spectrum}(A_c)$$

Sampled-data Model (*Remark* A_c *is invertible*):

$$\begin{cases} x_{k+1} = e^{A_c T}\, x_k + A_c^{-1}\,(e^{A_c T} - I)\, B_c\, u_k \\ y_k = C_c\, x_k \end{cases}$$

N. S. Cond. Of Asymptotic Stab.

$$|\mu_i| < 1 \qquad \forall\, \mu_i \in \text{Spectrum}(A)$$

We have the correspondance

$$\Re(\lambda_i) < 0 \qquad \Leftrightarrow \qquad |\mu_i| = |e^{\lambda_i T}| < 1$$

If the open-loop continuous-time system is asymptotically stable,Then the sampled-data system is also is asymptotically stable for all $T > 0$

IV.5 - Stability of Sampled-Data Systems: Closed-Loop

If the sampled-data system is in unitary feedback control structure, then

$$u_k = r_k - y_k = r_k - C_c x_k$$

where r_k is the reference signal. In closed loop, we have

$$\begin{cases} x_{k+1} = \mathcal{A}\, x_k + A_c^{-1}\,(e^{A_c T} - I)\, B_c\, r_k \\ y_k = C_c\, x_k \end{cases}$$

where

$$\mathcal{A} = e^{A_c T} - A_c^{-1}\,(e^{A_c T} - I)\, B_c C_c$$

The matrix $\mathcal{A}$ is sampling-period dependent. The eigenvalues of $\mathcal{A}$ are also sampling-period dependent and then stability property too.

IV.5 - Stability of Sampled-Data Systems: Closed-Loop

EXAMPLE

- Consider the continuous-time system

$$G(s) = \frac{K}{s(s+1)}$$

- If we put this system in unitary closed-feedback configuration, the closed-loop system is asymptotically stable for all $K > 0$ (Show it).
- The sampled-data transfer function $G(z)$ is *(see Chapter II)*

$$G(z) = \mathcal{Z}\left[B_0(s)\,\frac{K}{s(s+1)}\right] = \frac{z-1}{z}\,\mathcal{Z}\left[\frac{K}{s^2(s+1)}\right]$$

$$G(z) = K(e^{-T} - 1 + T)\,\frac{z-b}{(z-1)(z-e^{-T})}$$

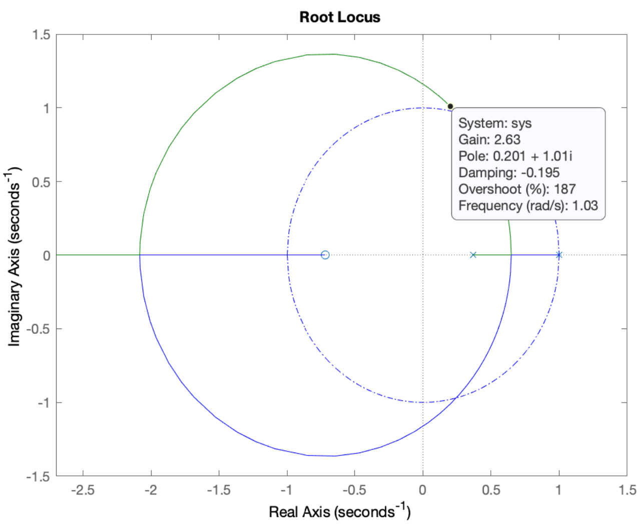
$$b = \frac{e^{-T}(T+1) - 1}{e^{-T} + T - 1}$$

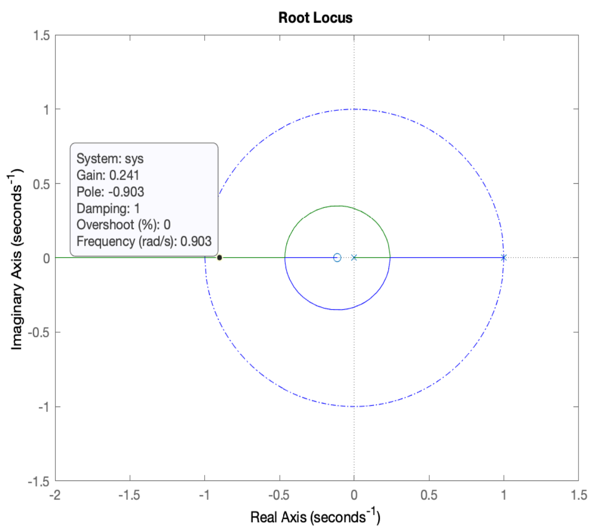
IV.5 - Stability of Sampled-Data Systems: Closed-Loop

EXAMPLE (Continued)

The characteristic polynomial is given by

$$1 + G(z) = 1 + \frac{K(e^{-T} - 1 + T)(z-b)}{(z-1)(z-e^{-T})} = 0$$





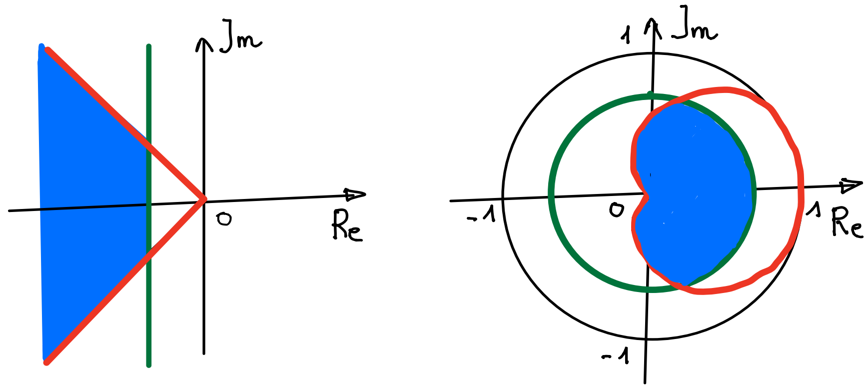
Root Locus for $T = 1\,s$ - Root Locus for $T = 10\,s$

Table of Contents I

IV.6 - Notion of Mode

- For continuous-time systems
  - Real poles $\lambda \rightarrow$ Aperiodic real mode
  - Complex poles $\lambda,\ \bar{\lambda} \rightarrow$ Oscillatory complex mode
- For discrete-time systems
  - Real poles $\lambda \rightarrow$ Real mode
    - ↗ Aperiodic if $\lambda > 0$
    - ↘ Oscillatory if $\lambda < 0$
  - Complex poles $\lambda,\ \bar{\lambda} \rightarrow$ Oscillatory complex mode

IV.6 - Notion of Mode

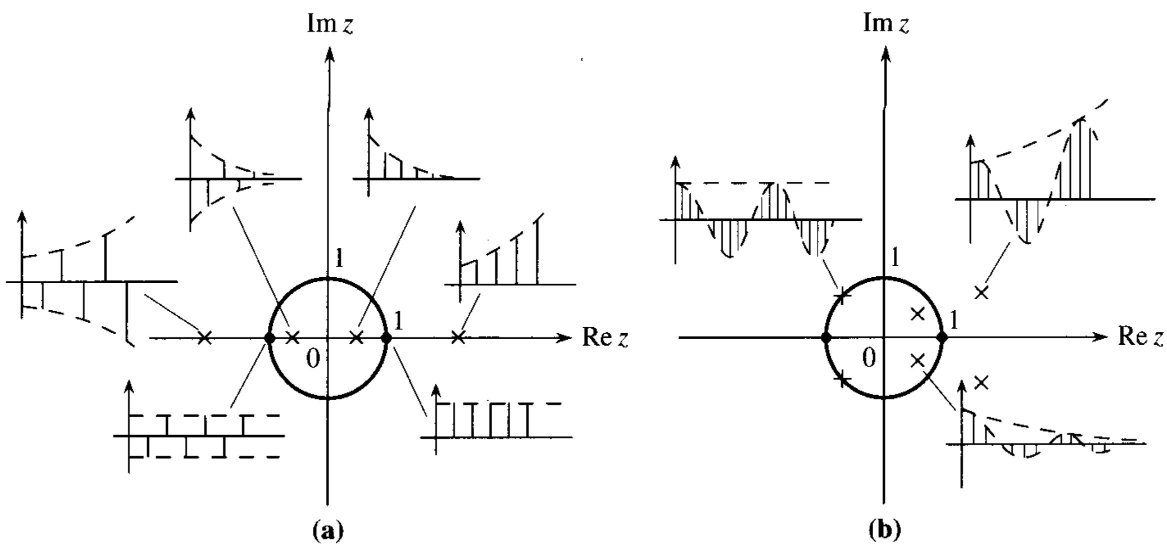


Continuous-time → s_i → $s_i = \alpha + j\omega$

Sampled-time → z_i → $z_i = e^{T\alpha} e^{jT\omega} = e^{T\alpha}(\cos T\omega + j \sin T\omega)$

$s_i \xrightarrow{e^{Ts_i}} z_i$

IV.6 - Notion of Mode



Time Responses of Poles: a) Real mode, b)Complex mode
(From C.T. Chen's Book)

DIGITAL CONTROL

Chapter V - Digital Implementation of Analog Compensators

The objectives of this chapter are

- To show how analog compensators can be used in the context of digital control
- To present some simple methods for a digital implementation of analog compensators

V.1 - Introduction

- In the context of linear time invariant systems, there exist several methods. to design simple compensators largely used in industry (lead-lag , PID compensators for example).
- The idea is to see how they can be simply adapted in the context of digital control.
- The principle consists in designing an analog compensator whose transfer function is $R_c(s)$.
- Derive a discrete-time compensator $R(z)$ which leads to a numerical algorithm implementable on a computer.
- A way consists in deriving approximated relations between Laplace variable s and variable z: $s = f(z)$. Then

$$R(z) = R_c(f(z)) = (R_c \circ f)(z)$$

V.2 - Forward Discretization

The derivative can be approximated as

$$\frac{dx}{dt} \approx \frac{x(t+T) - x(t)}{T}$$

Taking the Laplace transform, we have

$$sX(s) \approx \frac{\overbrace{e^{Ts}}^{z} - 1}{T} X(s)$$

Then, we obtain

$$\boxed{s \approx \frac{z-1}{T}}$$

This approximation can also be obtained remarking that

$$\boxed{\exp(Ts) \approx 1 + Ts}$$

V.3 - Backward Discretization

The derivative can also be approximated as

$$\frac{dx}{dt} \approx \frac{x(t) - x(t-T)}{T}$$

Taking the Laplace transform, we have

$$sX(s) \approx \frac{1 - \overbrace{e^{-Ts}}^{z^{-1}}}{T} X(s)$$

Then, we obtain *(Euler's Method)*

$$\boxed{s \approx \frac{1 - z^{-1}}{T} = \frac{z-1}{Tz}}$$

This approximation can also be obtained remarking that

$$\boxed{\exp(Ts) = \frac{1}{e^{-Ts}} \approx \frac{1}{1 - Ts}}$$

V.4 - Bilinear Approximation *(Tustin's approximation)*

If we consider

$$y(t) = \int_{\bullet}^{t} x(u)du = \int_{\bullet}^{t-T} x(u)du + \int_{t-T}^{T} x(u)du = y(t-T) + \int_{t-T}^{T} x(u)du$$

By *trapezoidal approximation* of the integral, we have

$$y(t) \approx y(t-T) + \frac{x(t-T)+x(t)}{2}\,T$$

Taking the Laplace transform, we have

$$Y(s) \approx e^{-Ts}\,Y(s) + \frac{e^{-Ts}+1}{2}\,T\,X(s) \Rightarrow Y(s) = \frac{1}{s}\,X(s) \approx \frac{T}{2}\,\frac{1+e^{-Ts}}{1-e^{-Ts}}\,X(s) = \frac{T}{2}\,\frac{1+z^{-1}}{1-z^{-1}}\,X(s)$$

And then

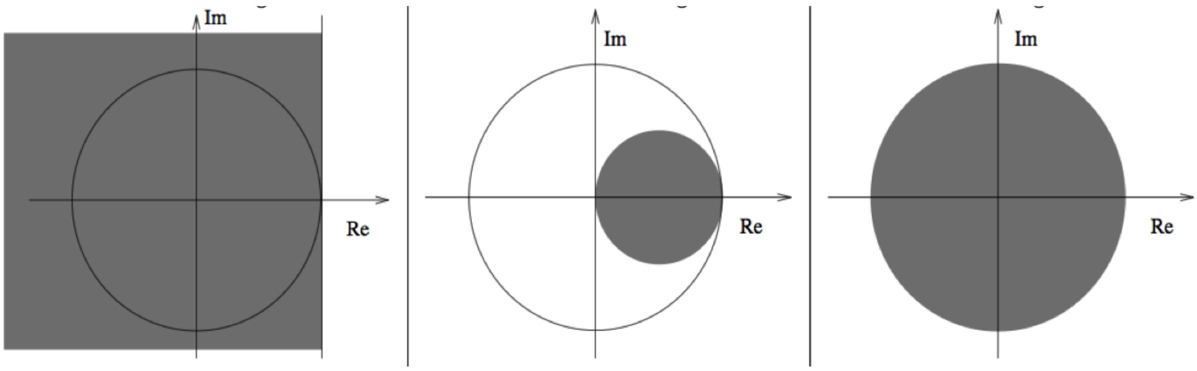
$$\boxed{s \approx \frac{2}{T}\,\frac{z-1}{z+1}}$$

it can also be obtained remarking that

$$\boxed{z = e^{Ts} = \frac{e^{Ts/2}}{e^{-Ts/2}} \approx \frac{1+Ts/2}{1-Ts/2}}$$

V.4 - Bilinear Approximation *(Tustin's approximation)*

The following figure shows how the stability region $\Re e(s) < 0$ in the s-plane is mapped onto the z-plane for the approximations above.



Forward Approximation Backward Approximation Bilinear Approximation

V.5 - Pole-Zero Matching Method

- The idea is to transform the poles and zeros of the compensator using the transformation

$$z = e^{Ts}$$

- Care is needed to preserve the static gain ($s = 0$) and the gain at high frequencies ($s \to \infty$). The static gain for discrete-time system is the gain for $z = 1$ and the gains at high frequencies is the gains for $z = -1$.

EXAMPLES

Consider the continuous-time compensator

$$R(s) = \frac{s+a}{s+b}$$

Applying the pole-zero matching method, the discrete-time compensator is

$$R(z) = \underbrace{\frac{a}{b}\frac{1-e^{-bT}}{1-e^{-aT}}}_{\text{To preserve DC- gain}} \; \frac{z-e^{-aT}}{z-e^{-bT}}$$

Remark that the high gain is not preserved for this example.

V.5 - Pole-Zero Matching Method

EXAMPLES (Continued)

Consider another continuous-time compensator

$$R(s) = \frac{s+a}{(s+b)(s+c)}$$

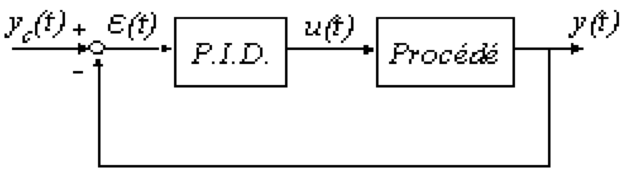
Applying the pole-zero matching method, the discrete-time compensator is

$$R(z) = \underbrace{\frac{a}{2bc}\frac{(1-e^{-bT})(1-e^{-cT})}{1-e^{-aT}}}_{\text{To preserve DC- gain}} \; \overbrace{(z+1)}^{\text{To preserve high-gain}} \; \frac{z-e^{-aT}}{(z-e^{-bT})(z-e^{-cT})}$$

The term $z+1$ is zero for $z=1$ and then the high gain is preserved because for the continuous-time regulator, the high gain is zero ($s \to \infty$).

V.6 - Digital PID Control

Consider the following closed-loop system



An analog PID can be written as

$$u(t) = k_p \varepsilon(t) + \frac{k_p}{\tau_i} \int_0^t \varepsilon(t) dt + k_p \tau_d \frac{d\varepsilon(t)}{dt}$$

Using backward-approximation, it can be numerically approximated by

$$u_k = k_p(\epsilon_k + \frac{T}{\tau_i} \sum_{j=0}^{k} \epsilon_j + \frac{\tau_d}{T}(\epsilon_k - \epsilon_{k-1})) = p_k + i_k + d_k$$

When the sampling period is small enough to invalidate the assumption of a negligible computing-time, ϵ_k can be simply predicted using the prediction $\hat{\epsilon}_k$ (linear extrapolation)

$$\hat{\epsilon}_k - \epsilon_{k-1} = \epsilon_{k-1} - \epsilon_{k-2} \to \hat{\epsilon} = 2\epsilon_{k-1} - \epsilon_{k-2}$$

V.6 - Digital PID Control: Implementation

The PID is tuned as done classically for the analog version. In general, some adaptations can be considered to obtain a control better adapted for real situations.

- The proportional term is taken as

$$p_{=}k_p(y_{ck} - y—k)$$

- The derivative part is

$$d_k = k_p \frac{\tau_d}{T}(-y_k + y_{k-1})$$

To limit the high frequency gain of the derivative term, the derivative $k_p \tau_d s$ is approximate by

$$\frac{k_p \tau_d s}{1 + s\tau_d/N}$$

and the differential equation giving the approximated analog derivative term is

$$\frac{\tau_d}{N}\frac{dD}{dt} + D = -k_p \tau_d \frac{dy}{dt}$$

V.6 - Digital PID Control: Implementation

A discretization by backward approximation leads to

$$d_k = \frac{\tau_d}{\tau_d + NT} d_{k-1} - \frac{k_p \tau_d N}{\tau_d + NT}(y_k - y_{k-1})$$

- The integral term is taken as

$$i_k = i_{k-1} + \frac{k_p T}{\tau_i}(y_{ck} - y_k) + \frac{T}{\tau_t}(u_{k-1} - v_{k-1})$$

where v_k is the input of the actuator and u_k its output. The signal u_k is measured and if a measure is not available, a model of actuator has to be included in the algorithm. τ_t is called *the tacking-time constant*.

The last term $\frac{T}{\tau_t}(u_{k-1} - v_{k-1})$ is only active when the actuator saturates. it can be seen as an additional loop which resets the integrator to an appropriate value with a time-constant τ_t.

V.6 - Digital PID Control: Implementation

An example of implementation with MATLAB

```
% yck: reference
% yk  measured output time k
% ykold: measured output time k-1
% dk: derivative term time k
% dkold: derivative term time k-1
% ik: integral term time k
% ikold: integral term time k-1
% T: sampling period
% uk: simulated output actuator (control)
% vk: PID output time k
% ukold, vkold: uk, vk time k-1

% Proportional term:
% kp: proportional gain

pk=kp*(yck-yk);

% Integral term:
% ti: Integration constant
% tt: tracking-time constant
ik=ikold+(kp*T/ti)*(yck-yk)+(T/tt)*(ukold-vkold);
```

```
% Derivative term:
% td: Derivation constant
% N: Maximal gain of derivative term

dk=(td/(td+N*T))*dkold-(kp*N*td/(td+N*T))*(yk-y

% Action P.I.D.:

vk=pk+ik+dk

% Simulated actuator:
% umin et umax: saturation limits

if vk<umin
        uk=umin;
elseif vk>umax
        uk=umax;
else
        uk=vk;
end
```


Introduction Forward Discretization *(Euler's Method)* Backward Discretization Bilinear Approximation *(Tustin's approximation)* Pole-Zero Matchi

V.7 - Zero-Order Hold

The techniques proposed above do not consider the presence of zero-order hold in the loop. To consider an approximated transfer function, we can remark that the zero-order hold introduces an average delay equal to $T/2$. In fact, we have

$$B_0(s) = \frac{1 - e^{-Ts}}{s} = e^{-Ts/2}\,\frac{e^{Ts/2} - e^{-Ts/2}}{s} \approx Te^{-\frac{Ts}{2}}$$

The method consists in applying the continuous-time design techniques using a modified open-loop transfer function

$$\boxed{G_o(s) = e^{-\frac{Ts}{2}}\,G(s)}$$

Introduction Forward Discretization *(Euler's Method)* Backward Discretization Bilinear Approximation *(Tustin's approximation)* Pole-Zero Matchi

V.8 - w-Transform Method

The method can be summarized as

1. Determination of the z-transfer function
$$G(z) = \mathcal{Z}(B_0(s)G_c(s))$$
2. Obtain the w-transfer function using
$$z = \frac{1+w}{1-w} \quad \longleftrightarrow \quad w = \frac{z-1}{z+1}$$
and
$$G(z) \longrightarrow G_m(w) = G_c\left(\frac{1+w}{1-w}\right)$$
3. Design a control law $R_m(w)$ using the model $G_m(w)$.
4. The digital control law is obtained by
$$R(z) = R_m\left(\frac{z-1}{z+1}\right)$$

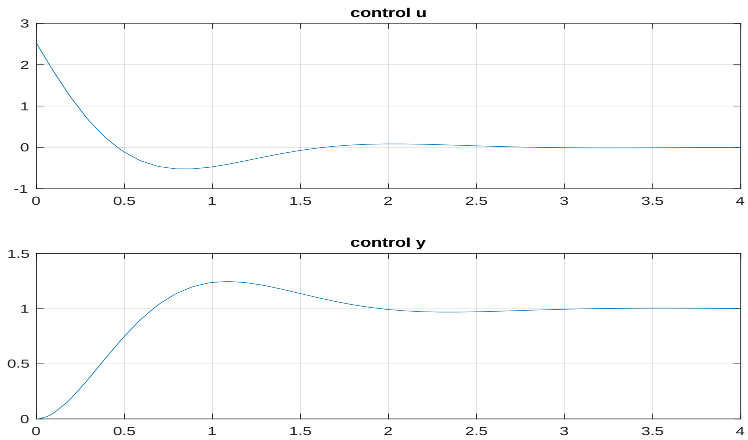
V.9 - Example

Consider the following system

$$G(s) = \frac{5}{s(s+1)}$$

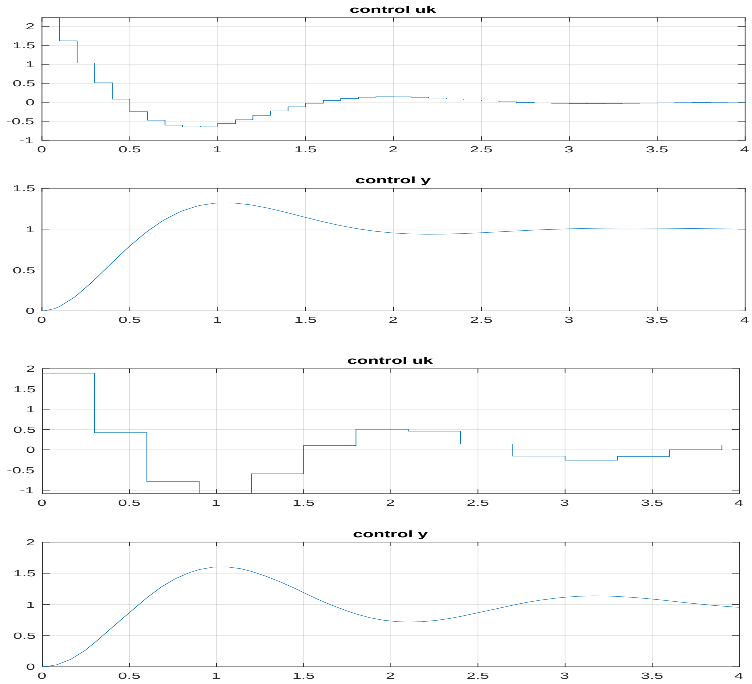
The gain has been selected equal to 5 to guarantee some steady-state performances (ramp). A phase lead compensator leading to a phase margin of 45 degrees is given by

$$R(s) = \frac{1 + 0,53s}{1 + 0,21s}$$



Method	Sampling Period $T = 0,1s$	Sampling Period $T = 0,3s$
Forward Method: $s = \frac{z-1}{T}$	$R(z) = \frac{2,52z - 2,05}{z - 0,52}$	$R(z) = \frac{2,52z - 1,09}{z + 0,43}$
Backward Method: $s = \frac{z-1}{zT}$	$R(z) = \frac{2,03z - 1,71}{z - 0,68}$	$R(z) = \frac{1,63z - 1,04}{z - 0,41}$
Tustin : $s = \frac{T}{2}\frac{z+1}{z-1}$	$R(z) = \frac{2,23z - 1,85}{z - 0,61}$	$R(z) = \frac{1,89z - 1,06}{z - 0,17}$
Matched pole-zero	$R(z) = \frac{2,20z - 1,8}{z - 0,62}$	$R(z) = \frac{1,76z - 0,99}{z - 0,24}$
Zero-order hold Approx.		$R(z) = \frac{3z - 1,8}{z + 0,2}$
w-Transform		$R(z) = \frac{2,72z - 1,57}{z + 0,15}$

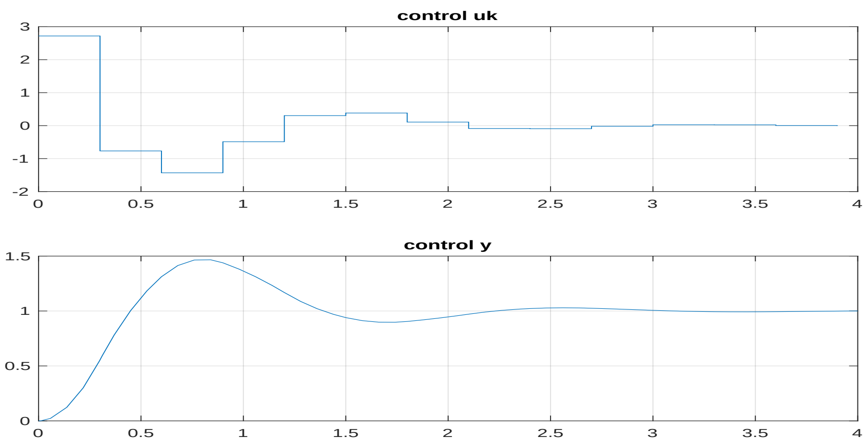
V.9 - Example: Tustin Approximation



Tustin Approximation: top: T=0.1s, bottom: T=0.3s

Introduction Forward Discretization *(Euler's Method)* Backward Discretization Bilinear Approximation *(Tustin's approximation)* Pole-Zero Matchi

V.9 - Example: Tustin Approximation



w-Transform Method: T=0.3s

DIGITAL CONTROL

Chapter VI - State-Space Control Design

The objectives of this chapter are

- Introduce the control design method when a discrete-time state-space model is available
- Consider the case where the state is measurable and develop the associated control, a state feedback control
- Discuss the case where only an output is measurable and present the state feedback/observer control structure

1. Introduction
2. State Feedback
3. State Reconstruction
4. Output Feedback Control: State-Feedback/Observer Control

VI.1 - Introduction

- When a state-space model is available, the state provides a complete knowledge about the system.
- It can be used for control purpose if it is a measurable quantity (if there exist the adequate sensors).
- If the state is not measurable, an output is generally measured and because there is fewer outputs than states, we call the problem *a partial information control*
- In that case, one possibility is to implement a state feedback where the state is replaced by an appropriate reconstructed signal

V.2 - State Feedback: Pole Placement

Consider the following system

$$\begin{cases} x_{k+1} = A\,x_k + B\,u_k \\ y_k = C\,x_k \end{cases}$$

A state feedback control is defined by

$$u_k = -L\,x_k + l_c\,y_{ck}$$

where y_{ck} is the reference signal, l_c is a gain used to impose a closed-loop static gain and $L = \begin{bmatrix} l_0 & l_1 & \cdots & l_{n-1} \end{bmatrix}$ is the state feedback gain. Then, the closed-loop system is

$$\begin{cases} x_{k+1} = (A - B\,L)x_k + Bl_c\,y_{ck} \\ y_k = C\,x_k \end{cases}$$

VI.2 - State Feedback: Pole Placement Fromm Controllability Form

Open-Loop System (Canonical Controllability Form)

The open-loop characteristic polynomial is

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

and the controllability canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \ddots & \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}'$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix}$$

VI.2 - State Feedback: Pole Placement From Controllability Form

Closed-Loop System

The desired closed-loop characteristic polynomial is

$$\Psi(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1} + z^n$$

and the closed-loop state-space model matrices are

$$A - BL = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \ddots & \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}$$

$$Bl_c = \begin{bmatrix} 0 & 0 & \cdots & 0 & l_c \end{bmatrix}'$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix}$$

where

$$\alpha_i = a_i + l_i \qquad i = 0, \ldots, n-1$$

VI.2 - State Feedback: Pole Placement From Controllability Form

State-Feedback and l_c Gains Computation

The control gain is given by

$$l_i = \alpha_i - a_i \qquad \forall i = 0 \cdots n-1$$

Closed-Loop transfer function

The closed-loop transfer function follows

$$\begin{aligned} G_F(z) &= \frac{l_c\,(b_0 + b_1\,z + \cdots + b_{n-1}\,z^{n-1})}{\alpha_0 + \alpha_1\,z + \cdots + \alpha_{n-1}\,z^{n-1} + z^n} \\ &= C\,(zI_n - A + B\,L)^{-1}\,B\,l_c \end{aligned}$$

and the gain l_c can be used to select an appropriate closed-loop static gain

$$\begin{aligned} G_F(1) &= \frac{l_c\,(b_0 + b_1\ + \cdots + b_{n-1}\ )}{\alpha_0 + \alpha_1\ + \cdots + \alpha_{n-1}\ + 1} \\ &= C\,(I_n - A + B\,L)^{-1}\,B\,l_c \end{aligned}$$

Introduction State Feedback State Reconstruction Output Feedback Control: State-Feedback/Observer Control

VI.2 - State Feedback: Pole Placement For Any Form

ALGORITHM

1. Open-Loop Characteristic Polynomial

$$P(z) = \det(zI_n - A) = z^n + a_{n-1}\,z^{n-1} + \cdots + a_1\,z + a_0$$

2. Feedback-gain for the controllability canonical form

$$\tilde{L} = [\tilde{l}_0 \quad \tilde{l}_1 \quad \cdots \quad \tilde{l}_{n-1}] \text{ with } \tilde{l}_i = \alpha_i - a_i \quad \forall i = 0, \cdots, n-1$$

3. Matrix M leading to the controllability canonical form

$$M = [m_1 \quad \cdots \quad m_n]$$

$$\begin{aligned} m_n &= B \\ m_{n-1} &= (A + a_{n-1}\,I_n)\,B \\ m_{n-2} &= (A^2 + a_{n-1}\,A + a_{n-2}\,I_n)\,B \\ &\cdots \\ m_1 &= (A^{n-1} + a_{n-1}\,A^{n-2} \cdots + a_1\,I_n)\,B \end{aligned}$$

4. Feedback-gain for the original system

$$L = \tilde{L}\,M^{-1}$$

VI.2 - State Feedback: Examples

EXAMPLE 1

Consider the system

$$F(s) = \frac{1}{s\,(1+s)}$$

The sampling period is $T = 1s$. The resulting transfer function $G(z)$ is

$$G(z) = \frac{0,3679z + 0,2642}{z^2 - 1,3679z + 0,3679}$$

and the controllability canonical form

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0 & 1 \\ -0,3679 & 1,3679 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 0,2642 & 0,3679 \end{bmatrix} x_k \end{cases}$$

The desired closed-loop dynamic is

$$P_{A-BL}(z) = z^2$$

Introduction State Feedback State Reconstruction Output Feedback Control: State-Feedback/Observer Control

VI.2 - State Feedback: Examples

EXAMPLE 1 (Continued)

The associated state-feedback gain is

$$L = [\alpha_0 - a_0 \quad \alpha_1 - a_1] = [-0,3679 \quad 1,3679]$$

and the closed-loop transfer function

$$\frac{Y(z)}{Y_c(z)} = G_F(z) = l_c \frac{0,3679z + 0,2642}{z^2}$$

To obtain a closed-loop static gain equal to 1, we impose

$$G(1) = 1 \Rightarrow 0,6321 \times l_c = 1 \Rightarrow l_c = 1,582$$

The state-feedback control law is

$$\boxed{u_k = -[-0,3679 \quad 1,3679]\, x_k + 1,582\, y_{ck}}$$

VI.2 - State Feedback: Examples

EXAMPLE 2

Consider a sampled system described by the state-space model

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0,55 & 0,12 \\ 0 & 0,67 \end{bmatrix} x_k + \begin{bmatrix} 0,01 \\ 0,16 \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \end{cases}$$

The characteristic polynomial is

$$P(z) = z^2 - 1,22\,z + 0,37 = z^2 + a_1\,z + a_0$$

Suppose that the desired closed-loop polynomial is (roots $0.3150 + 0.3328i - 0.3150 - 0.3328$)

$$\psi(z) = z^2 - 0,63\,z + 0,21 = z^2 + \alpha_1\,z + \alpha_0$$

The change of coordinate transforming the original state-space model into the controllability canonical form is given by

$$M = [A\,B + a_1\,B \quad B] = M = \begin{bmatrix} 0,0184 & 0,0100 \\ 0,0064 & 0,1600 \end{bmatrix}$$

VI.2 - State Feedback: Examples

EXAMPLE 2 (Continued)

The state feedback gain associated with the controllability canonical form is

$$\tilde{L} = [\alpha_0 - a_0 \quad \alpha_1 - a_1] = [-0,16 \quad 0,59]$$

and its expression for the original state-space model is

$$L = \tilde{L}\,M^{-1} = [9,22 \quad 3,11]$$

The resulting closed-loop static gain is

$$G_F(1) = C\,(I_2 - A + B\,L)^{-1}\,B\,l_c = 0,0388\,l_c$$

To have a static gain equal to 1, $l_c = 1/0,0388 = 25,78$. The complete control law becomes

$$\boxed{u_k = -[9,22 \quad 3,11]\,x_k + 25,78\,y_{ck}}$$

VI.2 - State Feedback: Examples

EXAMPLE 3

We can also proceed by direct identification. Consider the unstable system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

Find the state feedback gain $L = [l_0 \quad l_1]$ such that the eigenvalues of the closed-loop system are 1/2 and 1/4. We have

$$A - BL = A - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [l_0 \quad l_1] = \begin{bmatrix} 1 & 1 \\ -l_0 & 2 - l_1 \end{bmatrix}$$

Then, the closed-loop characteristic polynomial

$$P(\lambda) = \det(\lambda I - A + BL) = \lambda^2 + (l_1 - 3)\lambda + 2 + l_0 - l_1$$

By direct identification with

$$\psi(\lambda) = (\lambda - 1/2)(\lambda - 1/4) = \lambda^2 - 3/4\lambda + 1/8$$

we obtain

$$\boxed{L = [19/8 \quad 9/4]}$$

VI.2 - State Feedback: Structural condition

What are the structural conditions such that the pole placement problem be solvable?

- The problem is solvable if and only if the pair (A, B) is controllable
- A necessary and sufficient condition for controllability of pair (A, B) is (Kalman criterion)

$$\text{Rank}([B \ \ A^2B \ \ldots \ A^{n-1}B]) = \dim(x) = n$$

- If the system is not controllable, a stabilizing state-feedback exists if the system is *stabilizable*
- A pair (A, B) is stabilizable if the uncontrollable modes are asymptotically stable ($|\lambda| < 1$)

VI.2 - State Feedback: Structural condition

EXAMPLE

Consider the unstable system

$$x_{k+1} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_k$$

Find a state feedback $L = [l_0 \quad l_1]$ to place the closed-loop eigenvalues at 1/2 and 1/2. We have

$$A - BL = A - \begin{bmatrix} -1 \\ 1 \end{bmatrix} [l_0 \quad l_1] = \begin{bmatrix} l_0 + 2 & l_1 + 1 \\ -l_0 & 1 - l_1 \end{bmatrix}$$

Then the closed-loop characteristic polynomial is

$$\begin{aligned} P(\lambda) &= \det(\lambda I - A + BL) \\ &= \lambda^2 + (l_1 - l_0 - 3)\lambda + 2l_0 - 2l_1 + 2 = (\lambda - 2)(\lambda - l_0 + l_1 - 1) \end{aligned}$$

Identification with $\psi(\lambda) = (\lambda - 1/2)^2 = \lambda^2 - \lambda + 1/4$ is impossible. In fact, the system is not controllable. The eigenvalue 2 is not controllable *(Show it)*

$$[B \quad AB] = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

VI.2 - State Feedback: Matlab Procedure

- The function `L=place(A,B,lambda)` can be used to determine the gain L placing the closed-loop eigenvalues contained in the vector `lambda` if the pair (A, B) is controllable. For the example 2,

```
» A=[0.55 0.12; 0 0.67]; B=[0.01; 0.16];
» lambda=[0.3150 + 0.3328i - 0.3150 - 0.3328i];
» L=place(A,B,lambda)
```

- An important condition is that the closed-loop eigenvalues be selected different even if from a theoretically point of view, multiple eigenvalues can be selected.

VI.3 - State Reconstruction

When the state is not measurable, it is not possible to implement a state-feedback control law. In many situations, the only available information is an output. Consider the following system

$$\begin{cases} x_{k+1} &= A\,x_k + B\,u_k \\ y_k &= C\,x_k \end{cases}$$

The objective is to obtain an information about the state from the knowledge of the output y_k, the control u_k and the state-space model (A, B, C). The case $D \neq 0$ can be considered defining a new output $z_k = y_k - Du_k$. A dynamical system exploiting all the available information is given by

$$\begin{cases} \hat{x}_{k+1} &= \overbrace{A\,\hat{x}_k + B\,u_k}^{\text{model}} + \overbrace{H(y_k - \hat{y}_k)}^{\text{correction term}} \\ \hat{y}_k &= \underbrace{C\,\hat{x}_k}_{\text{estimated output}} \end{cases}$$

where the design parameter is the gain H. This systsem is called *a state reconstructor* or *an observer*. $\hat{x}_k$ is called the *the reconstructed state*.

VI.3 - State Reconstruction

To evaluate the performance of such a system, we define *the reconstruction error*

$$\epsilon_k = \hat{x}_k - x_k$$

Its dynamic can be written as

$$\begin{aligned} \epsilon_{k+1} &= \hat{x}_{k+1} - x_{k+1} \\ &= A\,\hat{x}_k + B\,u_k + HC(x_k - \hat{x}_k) - A\,x_k - B\,u_k \\ &= (A - LC)\epsilon_k \end{aligned}$$

If the initial error of reconstruction is ϵ_0, then we have

$$\epsilon_k = (A - LC)^k \epsilon_0$$

If all the eigenvalues λ_i of $A - LC$ are such that $|\lambda_i| < 1$

$$\lim_{k \to \infty} \epsilon_k = 0$$

and $\hat{x}_k \to x_k$ when $k \to \infty$. The state is asymptotically reconstructed.

VI.3 - State Reconstruction: From Observability Form

Observer

The open-loop characteristic polynomial is

$$P(z) = a_0 + a_1\,z + \cdots + a_{n-1}\,z^{n-1} + z^n$$

and the closed-loop state-space model matrices are

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & & \ddots & \ddots & \\ \vdots & & & \ddots & 0 \\ -a_1 & & & & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_1 & b_0 \end{bmatrix}'$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

VI.3 - State Reconstruction: From Observability Form

The desired closed-loop characteristic polynomial is

$$\phi(z) = \beta_0 + \beta_1 z + \cdots + \beta_{n-1} z^{n-1} + z^n$$

and the closed-loop state-space model matrices are

$$A - HC = \begin{bmatrix} -\beta_{n-1} & 1 & 0 & \cdots & 0 \\ -\beta_{n-2} & & \ddots & \ddots & \\ \vdots & & & \ddots & 0 \\ -\beta_1 & & & & 1 \\ -\beta_0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} h_{n-1} & h_{n-2} & \cdots & h_1 & h_0 \end{bmatrix} \quad (1)$$

where

$$\beta_i = a_i + h_i \qquad i = 0, \ldots, n-1$$

The observer gain H is easily obtained

$$\boxed{h_i = \beta_i - a_i \qquad \forall i = 0 \cdots n-1}$$

VI.3 - State Reconstruction: For Any Form

ALGORITHM

1. Open-Loop Characteristic Polynomial

$$P(z) = \det(zI_n - A) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

2. Observer-gain for the observability canonical form

$$\breve{H} = [\breve{h}_{n-1} \quad \cdots \quad \breve{h}_1 \quad \breve{h}_0]' \quad \text{with } \breve{h}_i = \beta_i - a_i \quad \forall i = 0, \cdots, n-1$$

3. Matrix M leading to the observability canonical form

$$M_o = ([m_1 \quad \cdots \quad m_n]')^{-1}$$

$$\begin{aligned} m_n &= C' \\ m_{n-1} &= (A' + a_{n-1} I_n) C' \\ m_{n-2} &= ((A')^2 + a_{n-1} A' + a_{n-2} I_n) C' \\ &\cdots \\ m_1 &= ((A')^{n-1} + a_{n-1} (A')^{n-2} \cdots + a_1 I_n) C' \end{aligned}$$

4. Observer-gain for the original system

$$H = M_o \breve{H}$$

VI.3 - State Reconstruction: For Any Form

EXAMPLE 2 (Previous Paragraph)

Consider the sampled system described by the state-space model ($T = 1s$)

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0,55 & 0,12 \\ 0 & 0,67 \end{bmatrix} x_k + \begin{bmatrix} 0,01 \\ 0,16 \end{bmatrix} u_k \\ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \end{cases}$$

The characteristic polynomial is

$$P(z) = z^2 - 1,22\,z + 0,37 = z^2 + a_1\,z + a_0$$

Suppose that the desired closed-loop polynomial is (roots 0.5, 0.6)

$$\varphi(z) = P_{A-HC}(z) = (z - 0,5)(z - 0,6) = z^2 - 1,1z + 0,3$$

The matrix of transformation M_o :

$$M_o = ([C' \qquad A'C' + a_1\,C']')^{-1}$$

$$M_o = \begin{bmatrix} 1 & 0 \\ 5,5833 & 8,3333 \end{bmatrix}$$

VI.3 - State Reconstruction: For Any Form

EXAMPLE 2 (Continued)

Observer gain for observation canonical form is given by

$$\breve{H} = [\beta_1 - a_1 \quad \beta_0 - a_0]' = [0,12 \qquad -0,0685]'$$

The gain for the orignal system is then

$$H = M_o\,\breve{H} = [0,12 \qquad 0,0992]'$$

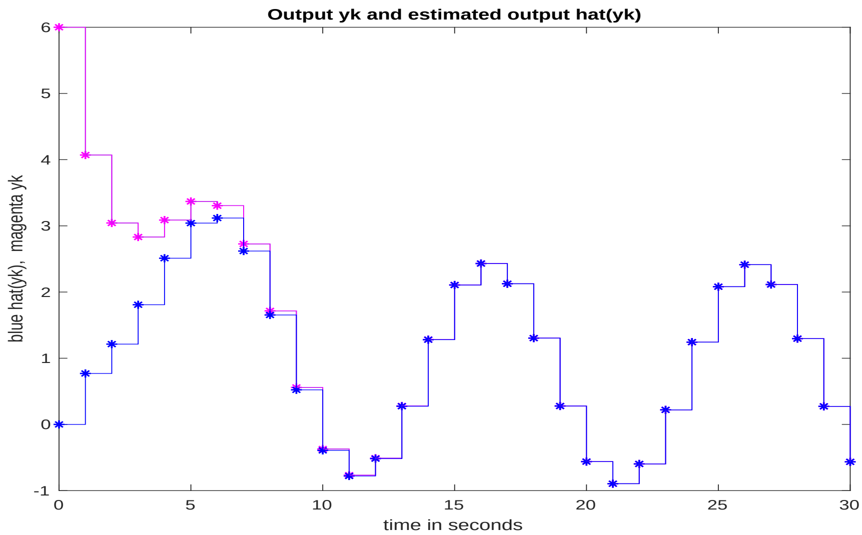
The observer equations are given by

$$\begin{cases} \hat{x}_{k+1} = \begin{bmatrix} 0,43 & 0,12 \\ -0,0992 & 0,67 \end{bmatrix} \hat{x}_k + \begin{bmatrix} 0,01 \\ 0,16 \end{bmatrix} u_k \\ \qquad + \begin{bmatrix} 0,12 \\ 0,0992 \end{bmatrix} y_k \\ \hat{y}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}_k \end{cases}$$

VI.3 - State Reconstruction: For Any Form

EXAMPLE 2 (Continued)

For an input given by $u_k = 5 + 30\sin(0.2\pi k)$, $T = 1s$ and initial condition $x_0 = [6 \;\; 0]'$, the following figure represents the output y_k and the estimated output $\hat{y}_k$.



Output y_k and estimated output $\hat{y}_k$

VI.4 - Output Feedback Control: State-Feedback/Observer Control

If only an output is measurable, the following control law can be implemented (state/feedback/observer control)

$$\begin{cases} x_{k+1} = A\,x_k + B\,u_k \\ y_k = C\,x_k \end{cases}$$

$$\begin{cases} \hat{x}_{k+1} = A\,\hat{x}_k + B\,u_k + H(y_k - C\hat{x}_k) \\ \epsilon_{k+1} = \hat{x}_{k+1} - x_{k+1} \end{cases}$$

$$u_k = -L\,\hat{x}_k + l_c y_{ck}$$

The closed-loop system is given by

$$\begin{bmatrix} x_{k+1} \\ \epsilon_{k+1} \end{bmatrix} = \begin{bmatrix} A - BL & -BL \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x_k \\ \epsilon_k \end{bmatrix} + \begin{bmatrix} Blc \\ 0 \end{bmatrix} y_{ck}$$

$$y_k = [C \quad 0] \begin{bmatrix} x_k \\ \epsilon_k \end{bmatrix}$$

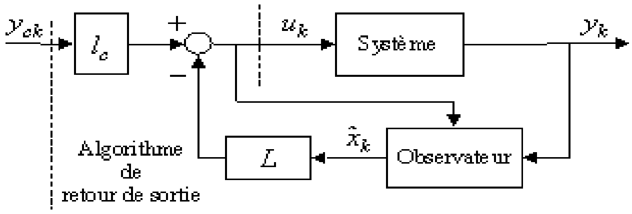
VI.4 - Output Feedback Control: State-Feedback/Observer Control

- The closed-loop system is of ordre $2n$ and the closed-loop poles are the poles of $(A - BL)$ et $(A - HC)$
- There is a separation, *the separation principle*, which *a priori* ensures that the design of the state-feedback can be done independently of the design the observer.
- A simple calculation shows that we have

$$\frac{Y(z)}{Y_c(z)} = C(zI_n - A + BL)^{-1} Bl_c$$

- The dynamic of the observer is not affected by the reference signal y_{ck} (uncontrollable). But the dynamic of the observer is observable from the output y_k.
- This suggests that the dynamic of the observer has to be selected faster than the state-feedback dynamic (dominance), in general between 3 and 10 times faster. But...

VI.4 - Output Feedback Control: Algorithm



$$\begin{cases} u_k = -L\,\hat{x}_k + l_c\,y_{ck} \\ \hat{x}_{k+1} = (A - HC)\hat{x}_k + B\,u_k + H\,y_k \end{cases}$$

Replacing u_k in the first equation, we have

$$\hat{x}_{k+1} = (A - HC - BL)\hat{x}_k + Bl_c\,y_{ck} + H\,y_k$$

Then

$$U(z) = -L[zI_n - (A-HC-BL)]^{-1}HY(z) + [1 - L[zI_n - (A-HC-BL)]^{-1}B]l_c\,Y_c(z)$$

And

$$U(z) = -R_1(z)Y(z) + R_2(z)Y_c(z)$$

From the previous equation, the algorithm (difference equation) involving y_{ck}, y_k and u_k can be deduced.

VI.4 - Output Feedback Control: State-Feedback/Observer Control

- Suppose that the real system can exactly described by

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u$$
$$y = (C + \Delta C)x$$

where ΔA, ΔB and ΔC represent model uncertainties.

- The closed-loop system in this case is described by

$$\dot{X} = \begin{bmatrix} (A+\Delta A) - (B+\Delta B)L & -(B+\Delta B)L \\ \Delta A - H\,\Delta C - \Delta B\,L & A - HC + \Delta B\,L \end{bmatrix} X + \begin{bmatrix} (B+\Delta B)l_c \\ -\Delta B\,l_c \end{bmatrix} y_c$$
$$y = \begin{bmatrix} C + \Delta C & 0 \end{bmatrix} X$$

The separation principle is not valid and we have to take into account the uncertainties.

- The idea is to derive a model for uncertainties
- Several approaches exist using norms of matrices ($\|\Delta A\| \leqslant \alpha$)
- Other approaches suppose specific forms for uncertainties (polytopic, norm-bounded...). See *Robust Control Literature*.

DIGITAL CONTROL

Chapter VII - RST Digital Control

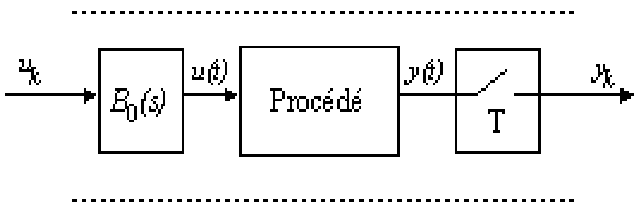
The objectives of this chapter are

- Introduce the RST digital control which is a two-degree-of-freedom control
- Show that under simple structural contraints, this control is adapted for solving a tracking control problem
- Give the main steps of the design of such a controller

1. Introduction
2. Tuning Method of RST digital Control
3. Regulation
4. Tracking
5. Example
6. Tracking and Regulation with Independent Objectives

VII.1 - Introduction

Consider the transfer function of a linear sampled-data system described in the following figure.

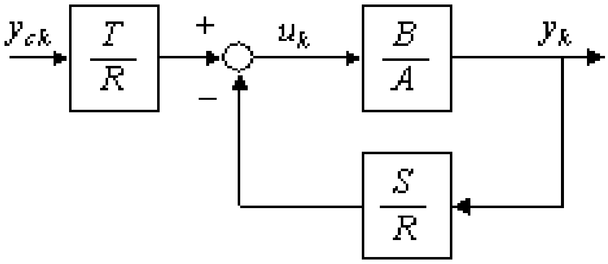


- The z-transfert function is given by

$$G(z) = \frac{B(z)}{A(z)}, \quad \deg(A) = n$$

- Recall that $A(z)$ and $B(z)$ are prime polynomials. If not, $G(z)$ is not the transfer function and the commun terms have to be simplified, revealing a lost of controllability or observability.

VII.1 - Introduction



The proposed algorithm is a two-degree-of-freedom control expressed as

$$R(z)\,U(z) = T(z)\,Y_c(z) - S(z)\,Y(z)$$

or

$$U(z) = -\frac{S(z)}{R(z)}Y(z) + \frac{T(z)}{R(z)}Y_c(z)$$

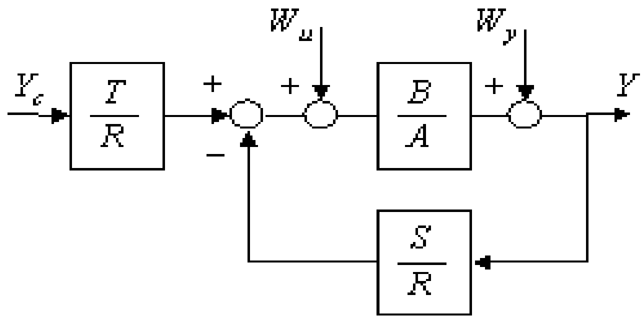
and

$$C_T(z) = \frac{T(z)}{R(z)} \text{ and } C_R(z) = \frac{S(z)}{R(z)}$$

$$\deg(R) \geqslant \deg(T), \quad \deg(R) \geqslant \deg(S)$$

VII.2 - Tuning Method of RST digital Control

We consider the following control structure



where W_u and W_y are respectively perturbations at input and at output.

The expression of the output is given by

$$Y(z) = \frac{B(z)T(z)}{A(z)R(z)+B(z)S(z)}Y_c + \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}W_u(z)$$
$$+\frac{A(z)R(z)}{A(z)R(z)+B(z)S(z)}W_y(z)$$

Introduction Tuning Method of RST digital Control Regulation Tracking Example Tracking and Regulation with Independent Objec

VII.2 - Tuning Method of RST digital Control

The tuning method is divided into the two following steps

i) *Regulation*: The objective is to reject the effects of the perturbations W_u and W_y controlling the dynamic of rejection. This can be done by pole placement using the degrees of freedom of $R(z)$ and $S(z)$

ii) *Tracking*: The objective is to track a signal reference y_{ck}. The last degree of freedom $T(z)$ can be used to satisfy the tracking objectives.

Important Remarks

- *In general, the dynamic of regulation is selected faster than the dynamic of tracking. This induces a time-decoupling between the two dynamics.*
- *If the controller in the loop $C_R(z)$ does not cancel zeros of $G(z)$, the zeros of $G(z)$ are invariant by feedback.*
- *The main limitation for the tracking control problem is related to the minimum phase property of open-loop system. Asymptotic tracking is possible for minimum phase systems.*

VII.3 - Regulation: Conditions of Rejection

Consider the perturbation (z-transform)

$$W(z) = \frac{N_W(z)}{D_W(z)}$$

- Remark that the closed-loop transfer functions involved in the reject of perturbations are

$$\frac{B(z)R(z)}{A(z)R(z) + B(z)S(z)} \text{and} \frac{A(z)R(z)}{A(z)R(z) + B(z)S(z)}$$

  then involving the controller $C_R(z)$ or polynomials $R(z)$ and $S(z)$.
- To solve the regulation problem, the role of $R(z)$ has to be investigated because it appears in the two numerators.
- The dynamic of regulation is determined by $A(z)R(z) + B(z)S(z)$. If $P(z)$ is the polynomial associated to the dynamic of regulation, under what generic conditions $R(z)$ and $S(z)$ satisfying the polynomial equation

$$A(z)R(z) + B(z)S(z) = P(z)$$

  exist?

VII.3 - Regulation: Conditions of Rejection

- The contribution of the perturbations on the output are described by

$$Y(z) = S_j(z)W(z) = \frac{N_j(z)}{P(z)}\frac{N_W(z)}{D_W(z)}, \ j = A, B$$

where $N_A(z) = A(z)R(z)$ and $N_B(z) = B(z)R(z)$.

- The perturbations will be rejected in steady state if the poles of $D_W(z)$ are compensated by the numerators $N_j(z), \ j = A, B$.
  - If the roots of $D_W(z)$ are not in the set of roots of $A(z)$ and $B(z)$, a way to reject the perturbations is to include them in the polynomial $R(z)$ associated with the controller.
  - In such a case, $R(z)$ is selected as

$$R(z) = D_W(z)R_1(z)$$

  - The conclusion is that for rejecting a perturbation, its model has to be present in the open-loop transfer function *(Internal Model Principle)*.

Introduction Tuning Method of RST digital Control Regulation Tracking Example Tracking and Regulation with Independent Objec

VII.3 - Regulation: Conditions of Rejection

EXAMPLES

- If the perturbation is a step signal (bias), we have

$$D_W(z) = z - 1$$

- If the perturbation signal is of *order* l

$$D_W(z) = (z - 1)^l$$

- If the perturbation is periodic, for example a cos or a sin

$$D_W(z) = z^2 - 2z\cos(\omega T) + 1$$

VII.3 - Regulation: Solving $P(z) = A(z)R(z) + B(z)S(z)$

The dynamic of regulation is defined by the polynomial

$$P(z) = A(z)R(z) + B(z)S(z)$$

where the degree of freedoms are $R(z)$, $S(z)$ and $P(z)$. Remark that the following properties are satisfied.

- If $A(z)$ and $B(z)$ have a common factor, the equation possesses a solution if the common factor divides $P(z)$.
- If $R_0(z)$ and $S_0(z)$ are solutions, then $R(z) = R_0(z) + Q(z)B(z)$ and $S(z) = S_0(z) - Q(z)A(z)$ are solutions for all arbitrary polynomial $Q(z)$. *(this can be verified by a direct substitution)*
- There is an infinite number of solutions, but there exists only one of minimal order such that

$$\deg(R) < \deg(B) \text{ or } \deg(S) < \deg(A)$$

VII.3 - Regulation: Solving $P(z) = A(z)R(z) + B(z)S(z)$

- Remark that if $\deg(S) \leqslant \deg(R)$ and $\deg(B) < \deg(A)$, we have

$$\deg(AR + BS) = \deg(AR) = \deg(P) \Rightarrow \boxed{\deg(R) = \deg(P) - \deg(A)}$$

Among the solutions $S(z)$ such that $\deg(S) < \deg(A)$, we select the one such that

$$\deg(S) = \deg(A) - 1$$

and the condition $\deg(R) \geqslant \deg(S)$ leads to

$$\deg(P) - \deg(A) \geqslant \deg(A) - 1 \Rightarrow \boxed{\deg(P) \geqslant 2\deg(A) - 1}$$

- Usually the polynomial $P(z)$ is selected to exhibit a dominant dynamic $P_{dom}(z)$ where $\deg(P_{dom}) < \deg(P)$. Then $P(z)$ is selected as

$$\boxed{P(z) = P_{dom}(z)P_{aux}(z)}$$

where $P_{aux}(z)$ is an auxiliary polynomial of appropriate dimension whose roots are non dominant with respect to the roots of $P_{dom}(z)$.

VII.3 - Regulation: Solving $P(z) = A(z)R(z) + B(z)S(z)$

The general procedure to solve the regulation problem

Step 1 - Select the dynamic of regulation

$$P(z) = P_{dom}(z)P_{aux}(z) \qquad \deg(P) \geqslant 2\deg(A) - 1$$

Step 2 - Solve the Diophantine equation

$$A(z)R(z) + B(z)S(z) = P(z)$$

where

$$\deg(R) = \deg(P) - \deg(A) \qquad \deg(S) = \deg(A) - 1$$

The Diophantine equation can be solved by the two equivalent techniques

- By direct substitution and identification of polynomials
- By solving a linear system of equations

VII.3 - Regulation: Solving $P(z) = A(z)R(z) + B(z)S(z)$

Consider a system of order n. The polynomials are explicitly defined by

$$\begin{aligned}
A(z) &= a_0 + a_1 z + a_2 z^2 + \cdot + a_n z^n \\
B(z) &= b_0 + b_1 z + b_2 z^2 + \cdot + b_n z^n \\
R(z) &= r_0 + r_1 z + r_2 z^2 + \cdot + r_{n-1} z^{n-1} \\
S(z) &= s_0 + s_1 z + s_2 z^2 + \cdot + s_{n-1} z^{n-1} \\
P(z) &= p_0 + p_1 z + p_2 z^2 + \cdot + p_{2n-1} z^{2n-1}
\end{aligned}$$

VII.3 - Regulation: Solving $P(z) = A(z)R(z) + B(z)S(z)$

The solution of the Diophantine equation is obtained solving the following linear system

$$\underbrace{\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & b_1 & a_0 & b_0 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & a_1 & b_1 & a_0 & b_0 & \cdots & 0 & 0 \\ a_3 & b_3 & a_2 & b_2 & a_1 & b_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & b_n & a_{n-1} & b_{n-1} & a_{n-2} & b_{n-2} & \cdots & a_1 & b_1 \\ 0 & 0 & a_n & b_n & a_{n-1} & b_{n-1} & \cdots & a_2 & b_2 \\ 0 & 0 & 0 & 0 & a_n & b_n & \cdots & a_3 & b_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_n & b_n \end{bmatrix}}_{\mathcal{M}} \underbrace{\begin{bmatrix} r_0 \\ s_0 \\ r_1 \\ s_1 \\ \vdots \\ \vdots \\ r_{n-2} \\ s_{n-2} \\ r_{n-1} \\ s_{n-1} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ p_{2n-4} \\ p_{2n-3} \\ p_{2n-2} \\ p_{2n-1} \end{bmatrix}}_{b}$$

The matrix $\mathcal{M}$ is a Sylvester matrix. $\mathcal{M}$ is invertible if the polynomials $A(z)$ and $B(z)$ are prime.

VII.3 - Regulation: Solving $P(z) = A(z)R(z) + B(z)S(z)$

When a reject of perturbation is explicitly taken into account, $R(z)$ is taken as

$$R(z) = D_W(z)R_1(z) \text{ with } \deg(D_W) = l$$

Adapted to this case, the previous approach leads to the following procedure

Step 1 - Select the dynamic of regulation

$$P(z) = P_{dom}(z)P_{aux}(z) \qquad \deg(P) \geqslant 2\deg(A) + l - 1$$

Step 2 - Solve the Diophantine equation

$$A(z)D_W(z)R_1(z) + B(z)S(z) = P(z)$$

where

$\deg(R_1) = \deg(P) - \deg(A) - l \qquad \deg(S) = \deg(A) + l - 1$

VII.4 - Tracking

In several practical problems, the objective is to track a specific reference signal y_{mk}. To formulate the problem in a general setting, it is possible to consider that the reference is the output of a model described by

$$Y_m(z) = G_m(z)Y_c(z) = \frac{B_m(z)}{A_m(z)}Y_c(z)$$

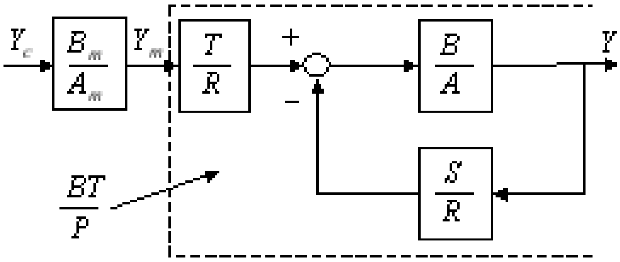
where $Y_c(z)$ is a normalized signal (impulse, step $\cdots$). $G_m(z)$ is called *the reference model* whose dynamic is in general slower than the dynamic of regulation. A classical example of such a model is given below

$$G_m(z) = \frac{b_{m0} + b_{m1}z}{a_{m0} + a_{m1}z + a_{m2}z^2}$$

This model can be obtained sampling a second order continuous-time system defined by damping ξ and frequency ω_n.

VII.4 - Tracking

With the model of reference the controlled system has the following structure



$T(z)$ is selected to satisfy the following conditions

- Obtain a unitary static gain between y_{mk} and y_k.
- Compensate totally or partially the dynamic of regulation (in general faster than the dynamic of tracking)

VII.4 - Tracking

Under these conditions $T(z)$ can be chosen as (if possible)

$$T(z) = P(z)T_1(z)$$

with

$$T_1(z) = \begin{cases} 1/B(1) & \text{if} \quad B(1) \neq 0 \\ 1 & \text{if} \quad B(1) = 0 \end{cases}$$

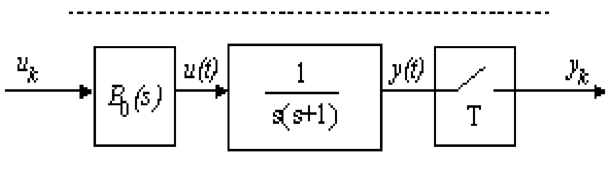
The output is expressed as

$$Y(z) = \frac{B_m(z)B(z)T_1(z)}{A_m(z)} Y_c(z)$$

If the dynamic of regulation is partially compensated, the non compensated part must be non dominant when compared to the dynamic of tracking $A_m(z)$.

VII.5 - Example

Consider the system



The sampled transfer-function was obtained and is given

$$G(z) = \frac{K(z-b)}{(z-1)(z-a)} = \frac{K(z-b)}{z^2-(a+1)z+a} = \frac{B(z)}{A(z)} \qquad (1)$$

with

$$K = e^{-T} - 1 + T \qquad a = e^{-T} \qquad b = 1 - \frac{T\left(1-e^{-T}\right)}{e^{-T}-1+T}$$

The dynamic of regulation is characterized by the dominant polynomial

$$P_{dom}(z) = z^2 + \alpha z + \beta$$

VII.5 - Example

The degrees of polynomials $P(z)$, $R(z)$ and $S(z)$ must satisfy

$$\deg(P) \geqslant 2\deg(A) - 1 = 3, \quad \deg(R) = \deg(P) - \deg(A) = 1, \quad \deg(S) = \deg(A) - 1 = 1$$

Polynomial $P(z)$ is taken of order 3. $z = 0$ is taken as non dominant pole and

$$P_{dom}(z)P_{aux}(z) = (z^2 + \alpha z + \beta)z = z^3 + \alpha z^2 + \beta z$$

Then the Diophantine equation is given by

$$(z^2 - (a+1)z + a)(r_1 z + r_0) + K(z - b)(s_1 z + s_0) = z^3 + \alpha z^2 + \beta z$$

and

$$r_1 z^3 + [Ks_1 - (a+1)r_1 + r_0]z^2 + [ar_1 - (a+1)r_0 - Kbs_1 + Ks_0)z + [ar_0 - Kbs_0] = z^3 + \alpha z^2 + \beta z$$

By identification of coefficients, we obtain

$$\begin{cases} r_1 &= 1 \\ -(a+1)r_1 + r_0 + Ks_1 &= \alpha \\ ar_1 - (a+1)r_0 - Kbs_1 + Ks_0 &= \beta \\ ar_0 - Kbs_0 &= 0 \end{cases}$$

VII.5 - Example: Continued

We can write the associated linear system

$$\begin{bmatrix} a & -Kb & 0 & 0 \\ -(a+1) & K & a & -Kb \\ 1 & 0 & -(a+1) & K \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \\ r_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \\ \alpha \\ 1 \end{bmatrix}$$

The solution can be obtained from computer algebra tools

$$r_0 = \frac{b(b - a + ab + \beta + b\alpha)}{(a-b)(1-b)}, \quad r_1 = 1$$

$$s_0 = \frac{a(b - a + ab + \beta + b\alpha)}{K(a-b)(1-b)}, \quad s_1 = \frac{a - b - ab - a^2 b + a^2 + (a-b)\alpha - b\beta - ab\alpha}{K(a-b)(1-b)}$$

The polynomial $T(z)$ can be selected to compensate a part of the dynamic of regulation and obtain a unitary static gain

VII.5 - Example: Continued

- The sampling period T has been selected equal to $0.1s$. The open-loop z-transfer function is

$$G(z) = \frac{B(z)}{A(z)} = \frac{0,0048z + 0,0047}{z^2 - 1,9048z + 0,9048}$$

- For the regulation, the dominant dynamic is associated with a second order polynomial defined by parameters $\zeta = 0,45$ and $\omega_n = 5rd/s$ leading to the polynomial

$$z^2 - 1,4405z + 0,6376 = 0$$

- The characteristic of the tracking dynamic is associated with a second order polynomial defines by parameters $\zeta = 0,7$ and $\omega_n = 3rd/s$ leading to the reference model

$$\frac{B_m(z)}{A_m(z)} = \frac{0.0729}{z^2 - 1,5841z + 0,6570}$$

VII.5 - Example: Continued

With the numerical values above $K = 0.0048$, $a = 0.9048$, $b = -0.9672$, $\alpha = -1.4405$ and $\beta = 0.6376$. Then

$$R(z) = z + 0.1881, \quad S(z) = 57.10\, z - 36.38$$

and the polynomial $T(z)$ can be chosen to compensate the pole $z = 0$ and guaranteeing a unitary static gain

$$T(z) = \frac{P(1)z}{B(1)} = 20.74\, z$$

The controllers associated with the previous polynomials are

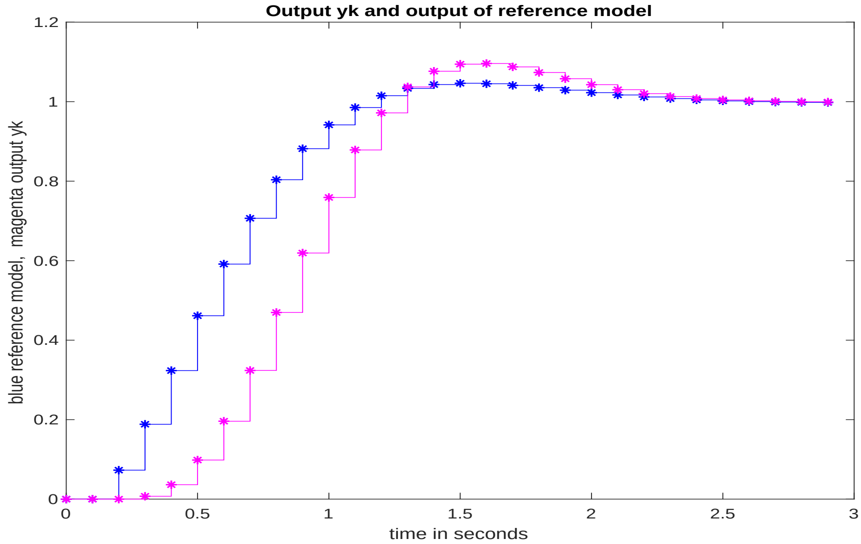
$$\boxed{\frac{S(z)}{R(z)} = \frac{57.10\, z - 36.38}{z + 0.1881}, \quad \frac{T(z)}{R(z)} = \frac{20.74\, z}{z + 0.1881}}$$

The closed-loop transfer function is

$$\boxed{G_{BF}(z) = \frac{1,5120\, z\, (0,0048\, z + 0,0047)}{(z^2 - 1,4405z + 0,6376)(z^2 - 1,5841\, z + 0,6570)}}$$

VII.5 - Example: Continued

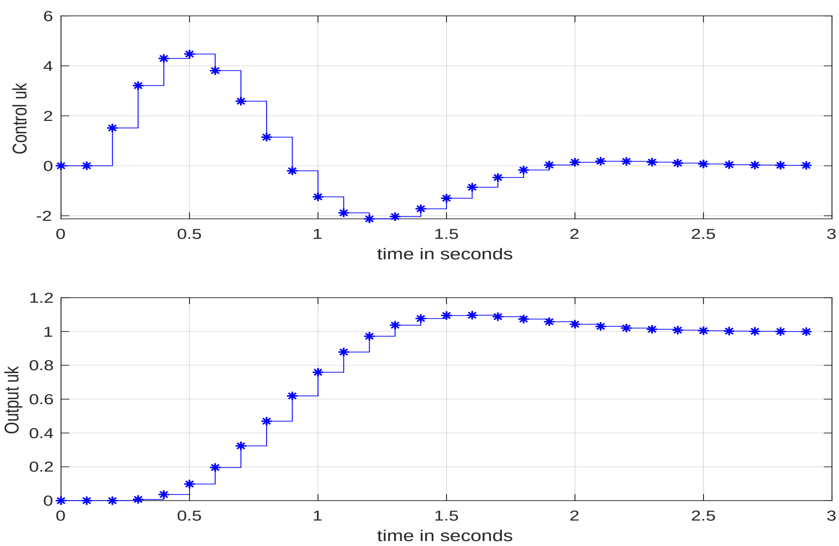
For the previous RST control, the system output is compared to the output of reference model.



Magenta: System Output, Blue: Reference Model Output

VII.5 - Example: Continued

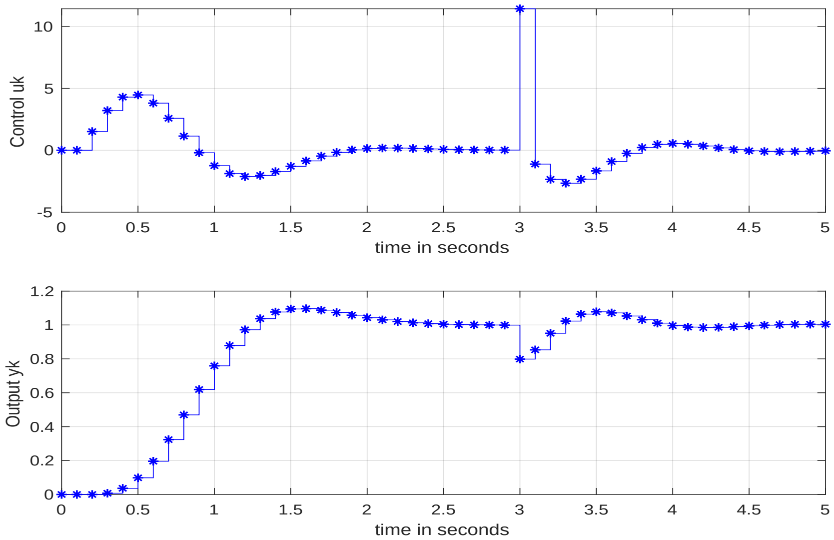
For the previous RST control, the control and the system output are represented in the following figure



Control u_k and System Output y_k

VII.5 - Example: Continued

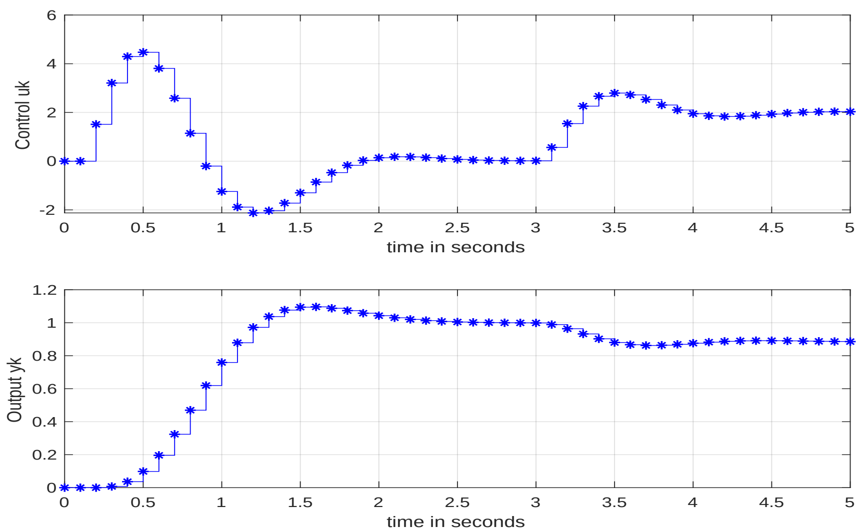
If a perturbation (step) affects the system output, it is rejected as shown in the following figure because polynomial $A(z)$ contains a term $(z-1)$.



Rejected additive step perturbation of amplitude -0.2 affecting the output

VII.5 - Example: Continued

But a step perturbation affecting the system input is not rejected because the polynomials $B(z)$ and $R(z)$ do not contain a term $(z-1)$.



Additive step perturbation of amplitude -2 affecting the system input

VII.5 - Example: Continued

In order to reject a step perturbation at the input of the system, a new RST-controller is designed introducing in the polynomial $R(z)$ the denominator of the perturbation, i.e. $D_W(z) = z - 1$ and

$$R(z) = (z-1)R_1(z)$$

In that case, the polynomials $P(z)$, $R(z)$ et $S(z)$ must satisfy ($l = 1$)

$$\deg(P) \geqslant 2\deg(A) + l - 1 = 4, \qquad \deg(R_1) = \deg(P) - \deg(A) - l = 1$$

$$\deg(S) = \deg(A) + l - 1 = 2$$

Then

$$R(z) = r_0 + r_1\, z$$

$$S(z) = s_0 + s_1\, z + s_2\, z^2$$

$T(z)$ is chosen to maintain a unitary static gain and compensate the term of the dynamic of regulation $z^2 - 1,4405z + 0,6376$

$$T(z) = \frac{z^2 - 1,4405z + 0,6376}{B(1)}$$

VII.5 - Example: Continued

To solve the associated Diophantine equation is equivalent to solve the following linear system (*Show it*)

$$\begin{bmatrix} -a_0 & b_0 & 0 & 0 & 0 \\ a_0 - a_1 & b_1 & -a_0 & b_0 & 0 \\ a_1 - a_2 & b_2 & a_1 - a_1 & b_1 & b_0 \\ a_1 & 0 & a_1 - a_2 & b_2 & b_1 \\ 0 & 0 & a_2 & 0 & b_2 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \\ r_1 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta \\ \alpha \\ 1 \end{bmatrix}$$

Solving all the previous equations leads to the RST-controller

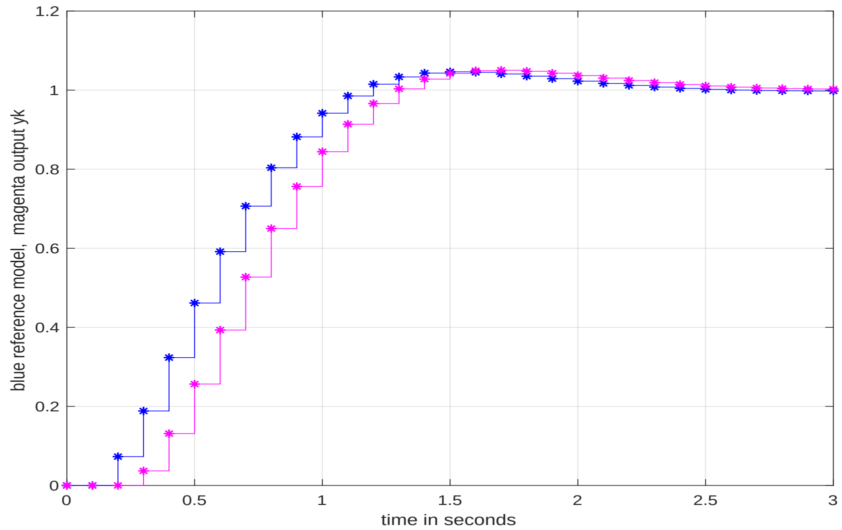
$$\boxed{\frac{S(z)}{R(z)} = \frac{182.45z^2 - 275.00z + 113.20}{(z-1)(z+0.59)}, \quad \frac{T(z)}{R(z)} = \frac{105.26z^2 - 151.63z + 67.11}{(z-1)(z+0.59)}}$$

The closed-loop transfer function is

$$\boxed{G_{BF}(z) = \frac{7,6737(0,0048\, z + 0,0047)}{z^2 - 1,5841z + 0,657}}$$

VII.5 - Example: Continued

For the previous RST control, the system output is compared to the output of reference model.

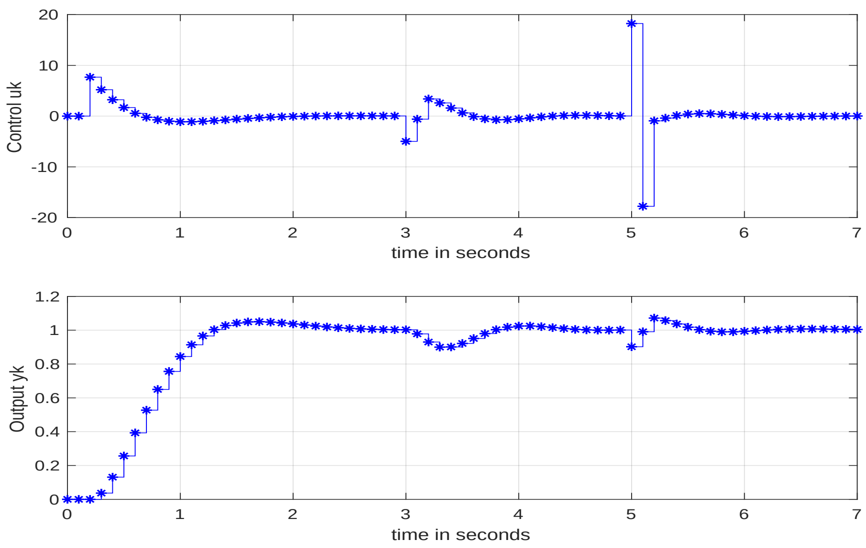


Magenta: System Output, Blue: Reference Model Output

Introduction Tuning Method of RST digital Control Regulation Tracking Example Tracking and Regulation with Independent Objec

VII.5 - Example: Continued

If perturbations (steps) affects the input and the output, now both are rejected as shown in the following figure.



Rejected additive step perturbation of amplitude -0.1 affecting the output at time $t = 5s$ and additive step perturbation of amplitude -5 affecting the input at time $t = 3s$

1 Introduction

2 Tuning Method of RST digital Control

3 Regulation

4 Tracking

5 Example

6 Tracking and Regulation with Independent Objectives

VII.6 - Tracking and Regulation with Independent Objectives

For plant with stable zeros, it is possible to use a strategy called *tracking and regulation with independent objectives*.

The closed-loop transfer function is given

$$G_{BF}(z) = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)}$$

Then it is possible to include $B(z)$ in the polynomial $R(z)$. $R(z)$ is selected as

$$R(z) = B(z)R_1(z)$$

Replacing in the closed-loop transfer $G_{BF}(z)$, we have

$$G_{BF}(z) = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} = \frac{B(z)T(z)}{B(z)\ [A(z)R_1(z) + S(z)]} = \frac{B(z)T(z)}{B(z)P(z)} = \frac{T(z)}{P(z)}$$

VII.6 - Tracking and Regulation with Independent Objectives

If in the objectives, we add the rejection of a perturbation whose denominator is $D_W(z)$, $\deg(D_W) = l$, $R(z)$ is selected as

$$R(z) = D_W(z)B(z)R_1(z)$$

Step 1 - Select the dynamic of regulation

$$P(z) = P_{dom}(z)P_{aux}(z) \qquad \deg(P) \geqslant 2\deg(A) - \deg(B) + l - 1$$

Step 2 - Solve the Diophantine equation

$$A(z)D_W(z)R_1(z) + S(z) = P(z)$$

where

$$\deg(R_1) = \deg(P) - \deg(A) - l \qquad \deg S = \deg(A) + l - 1$$

VII.6 - Tracking and Regulation with Independent Objectives

EXAMPLE (Continued)

Recall the main objectives for the example. The open-loop transfer function is

$$G(z) = \frac{B(z)}{A(z)} = \frac{0,0048z + 0,0047}{z^2 - 1,9048z + 0,9048}$$

The dynamic of regulation

$$z^2 - 1,4405z + 0,6376 = 0$$

and the reference model

$$\frac{B_m(z)}{A_m(z)} = \frac{0.0729}{z^2 - 1,5841z + 0,6570}$$

A term $z - 1$ being present in $A(z)$, an additive step perturbation is rejected at the output. To reject an additive perturbation at the input, a term $z - 1$ has to be included in $R(z)$.

VII.6 - Tracking and Regulation with Independent Objectives

EXAMPLE (Continued)

To apply the method of tracking and regulation with independent objectives, $R(z)$ will be given by

$$R(z) = (z-1)B(z)R_1(z)$$

The degrees of polynomials will be

$$\deg(P) = 2\deg(A) - \deg(B) + l - 1 = 4 - 1 + 1 - 1 = 3$$

$$\deg(R_1) = \deg(P) - \deg(A) - l = 3 - 2 - 1 = 0 \Rightarrow R_1(z) = 1$$

$$\deg S = \deg(A) + l - 1 = 2 + 1 - 1$$

The diophantine equation is

$$A(z)(z-1)S(z) = zP(z) \Rightarrow S(z) = zP(z) - A(z)(z-1) = 1.4643\, z^2 - 2.1720\, z + 0.9048$$

$T(z) = z^2 - 1,4405z + 0,6376$ is chosen to compensate the dynamic of regulation.

VII.6 - Tracking and Regulation with Independent Objectives

EXAMPLE (Continued)

The RST controller is defined as

$$\frac{S(z)}{R(z)} = \frac{1.4643z^2 - 2.1720\, z + 0.9048}{(z-1)(0,0048z + 0,0047)}, \quad \frac{T(z)}{R(z)} = \frac{z^2 - 1,4405z + 0,6376}{(z-1)(0,0048z + 0,0047)}$$

Then the closed-loop transfer function is

$$G_{BF}(z) = \frac{B_m(z)}{z\, A_m(z)} = \frac{0.0729}{z\,(z^2 - 1,5841z + 0,6570)}$$

The control algorithm is given by

$$A_m(z)R(z)U(z) = B_m(z)T(z)Y_c(z) - A_m(z)S(z)Y(z)$$

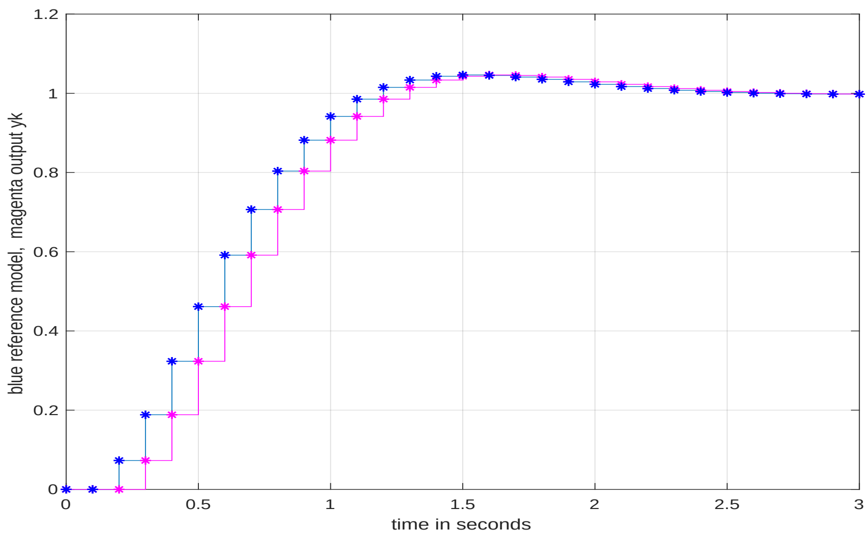
and

$$\begin{gathered} 0.0048u_{k+4} - 0.0077u_{k+3} - 0.0014u_{k+2} + 0.0074u_{k+1} - 0.0031u_k = \\ 0.0729y_{c_{k+2}} - 0.1050y_{c_{k+1}} + 0.0465y_{c_k} - 1.4643y_{k+4} \\ +4.4916y_{k+3} - 5.3075y_{k+2} + 2.8603y_{k+1} - 0.5945y_k \end{gathered}$$

VII.6 - Tracking and Regulation with Independent Objectives

EXAMPLE (Continued)

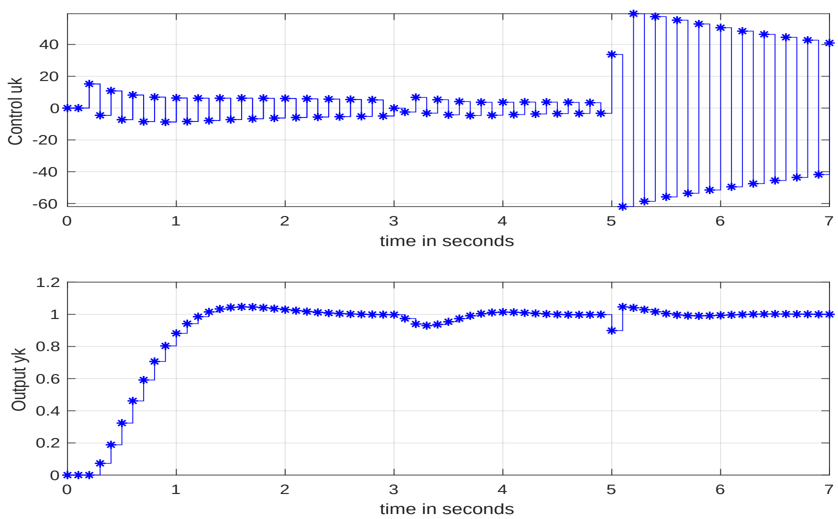
For the previous RST control, the system output is compared to the output of reference model.



Magenta: System Output, Blue: Reference Model Output

VII.6 - Tracking and Regulation with Independent Objectives

If perturbations (steps) affects the input and the output, both are rejected as shown in the following figure. There is a problem of sensitivity of the input to output perturbation.



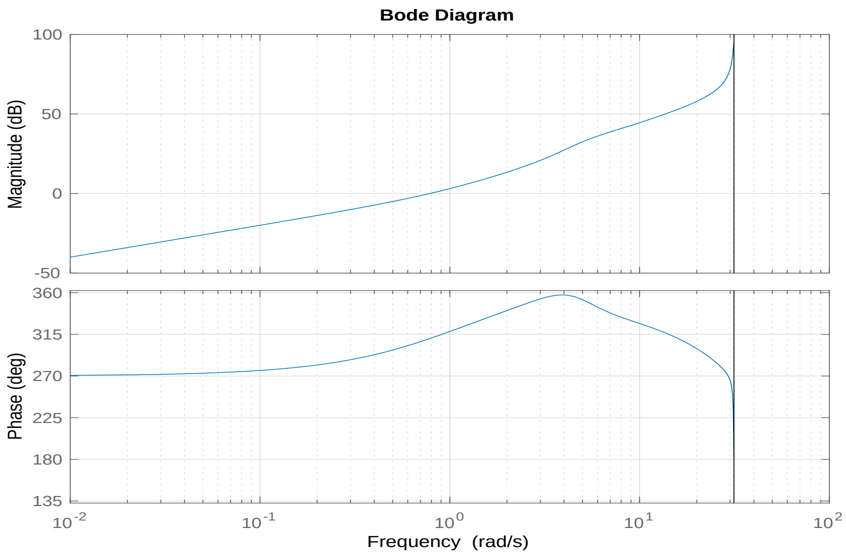
Rejected additive step perturbation of amplitude -0.1 affecting the output at time $t = 5s$ and additive step perturbation of amplitude -5 affecting the input at time $t = 3s$

VII.6 - Tracking and Regulation with Independent Objectives

The sensitivity function between the output perturbation and input is given by *(Show it)*

$$S_{yu}(z) = \frac{W_y(z)}{U(z)} = \frac{-A(z)S(z)}{A(z)R(z)+B(z)S(z)}$$

$$= 10^2 \frac{-1.46z^4 + 4.96z^3 - 6.37z^2 + 3.69z - 0.82}{z(0.48z^3 - 0.22z^2 - 0.37z + 0.30)}$$



Bode plot of $S_{yu}(z)$