MULTIVARIABLE SYSTEMS

(Course Material)

4 AE-SE

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE

LAAS CNRS

MULTIVARIABLE SYSTEMS

- **LECTURE**: *10 Sessions (11h30)* - Germain GARCIA
- **TUTORIALS and PRACTICAL WORK**: *7 Sessions (8h45)* - Elodie CHANTHERY and Audine SUBIAS
- **EXAMS**: *Online exam with moodle (duration: about 1h15): questions about lecture topics, tutorials, practical work and course application*
- **PREREQUISITES**: Linear Algebra, Analysis and Control of Single-Input, Single-Output (SISO) Systems
- **MOODLE** : *these slides, lecture notes and complementary documents associated with tutorials and practical work can be downloaded from the MOODLE platform*

MULTIVARIABLE SYSTEMS

Chapter I

Introductory Chapter

Objective of Chapter I

- Introduce the course from an example
- Show the difficulties to design a control law in the multi-input case
- Give a list of references

Outline of Chapter I

I-1. Stabilization of a lateral aircraft motion

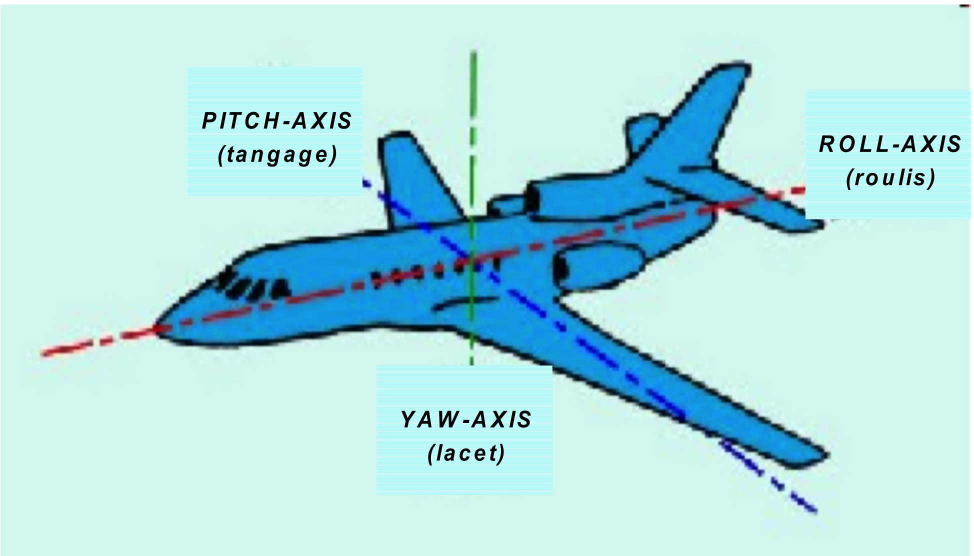
I-2. Analysis in Open Loop

I-3. Analysis of a Monovariable Design

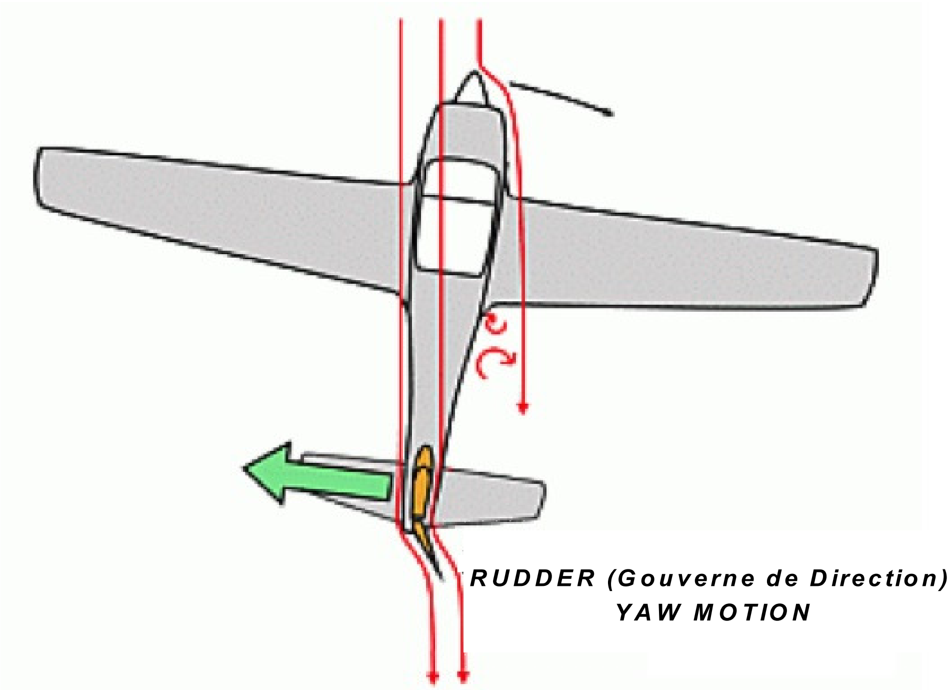
I-4. Outline of the Course

I-5. Some references

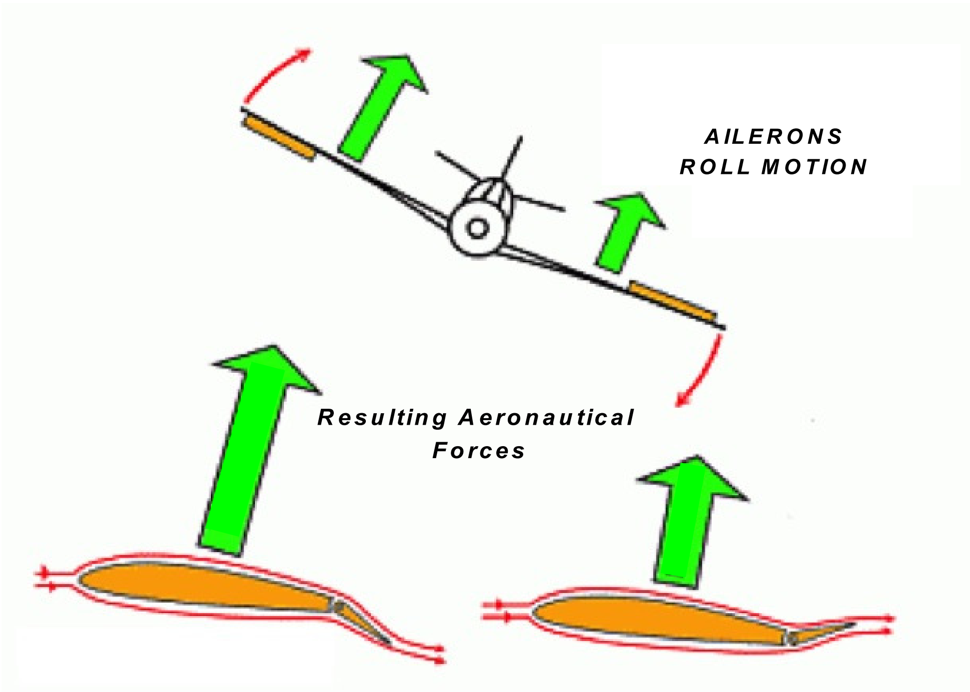
I.1) Stabilization of a lateral aircraft motion



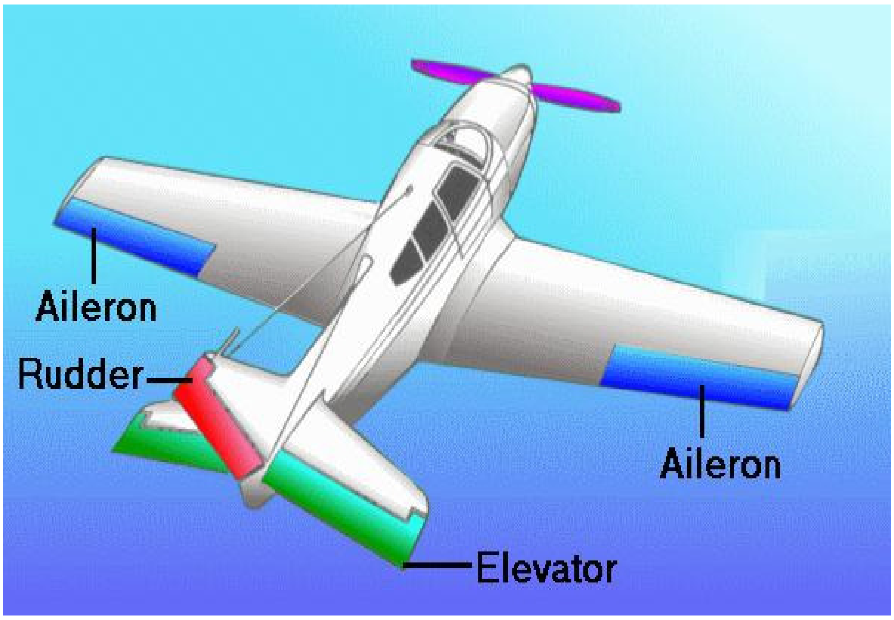
The six degrees of freedom: forward/back, up/down, left/right, pitch, yaw, roll



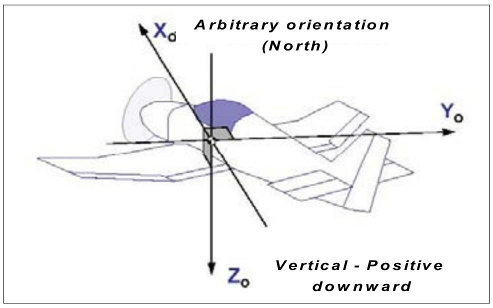
The rudder allows the control of the yaw motion



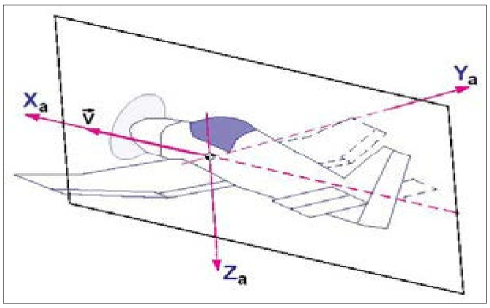
The ailerons allow the control of the roll motion



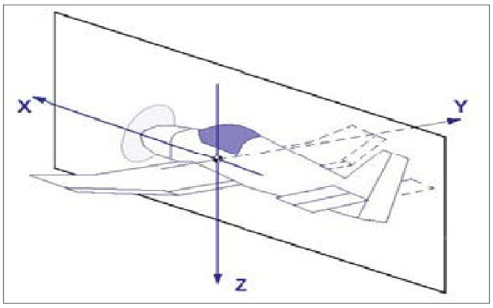
The elevators allow the control of the pitch motion



Inertial Coordinate System



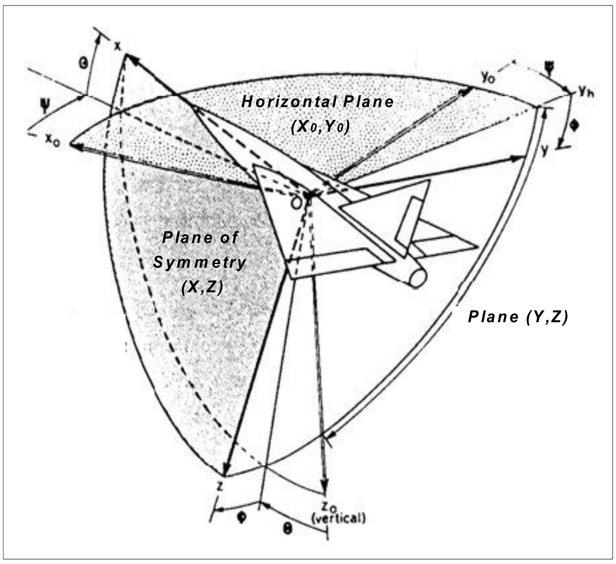
Aeronautical Coordinate System



Aircraft Coordinate System

Three frames of reference can be defined to derive a model for the aircraft

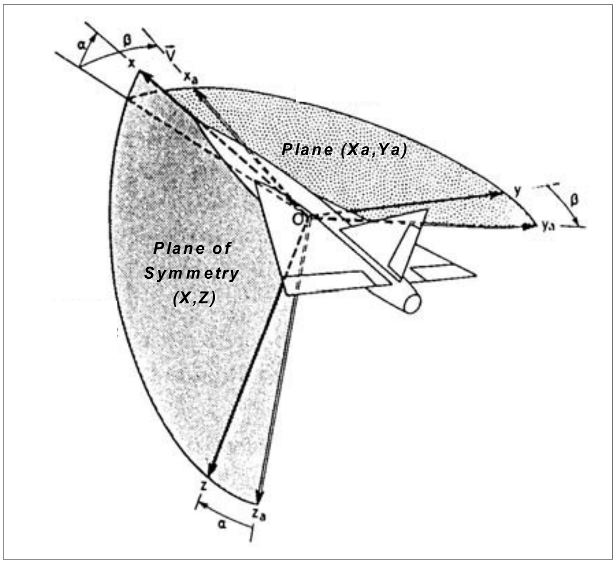
It is important to know precisely the relations between the coordinates of the aircraft in all the frames of reference (Euler angles). The use of a frame depends of the objectives.



- Ψ: Yaw Angle *(lacet)*, rotation axis Oz_0
- Θ: Pitch Angle *(tangage)*, rotation axis Oy_0
- Φ: Roll Angle *(roulis)*, rotation axis Ox_0

Relations between the aircraft coordinate system and the inertial frame when this last one is attached to the plane.

When the axis Oz_a of the aeronautical frame of reference belongs to the plane of symmetry of the aircraft two angles are used to relate aeronautical frame to aircraft one.



- α: Angle of Attack *(Angle d'Incidence)*, Angle between Ox and the plane (x_a, y_a)
- β: Side-Slip Angle *(Angle de dérapage)*, Angle between the plane of symmetry and velocity vector Ox_a

We apply the mechanical newton laws of motion in an inertial (galilean) frame of reference

$$\frac{d(m\overrightarrow{V})}{dt} = \sum_i \overrightarrow{F_{ei}}$$

$$\frac{d\overrightarrow{C}}{dt} = \sum_i \overrightarrow{M_{ei}}$$

$\overrightarrow{F_{ei}}$: External forces, Thrust of jet engines *(poussée des réacteurs)*,weight *(poids)*, lift *(portance)* and drag *(traînée)* forces

m, $\overrightarrow{V}$: Aircraft mass and Velocity

$\overrightarrow{M_{ei}}$: Moments due to external forces

$\overrightarrow{C}$: Angular Momentum *(moment cinétique)*

At the equilibrium,

$$\frac{d(m\overrightarrow{V})}{dt} = 0 \text{ and } \frac{d\overrightarrow{C}}{dt} = 0$$

and then

$$\sum_i \overrightarrow{F_{ei0}} = 0 \text{ and } \sum_i \overrightarrow{M_{ei0}} = 0$$
$$m\overrightarrow{V} \text{ and } \overrightarrow{C} \text{ are constant}$$

The equilibrium forces are the weight, the thrust, the drag and lift forces.

If we decompose the forces and moments as

$$\overrightarrow{F_{ei0}} + \Delta\overrightarrow{F_{ei0}} \quad \text{and} \quad \overrightarrow{M_{ei0}} + \Delta\overrightarrow{M_{ei0}}$$

The laws of motion become

$$\frac{d(m\overrightarrow{V})}{dt} = \sum_i \Delta\overrightarrow{F_{ei0}}$$

$$\frac{d\overrightarrow{C}}{dt} = \sum_i \Delta\overrightarrow{M_{ei0}}$$

The main adopted assumptions are

- The earth *(Terre)* is an inertial frame of reference fixed in space
- The aircraft mass m is constant
- The aircraft is a rigid body. In consequence, translation and rotation motions can be fully described with respect to the center of gravity
- The aircraft has a plane of symmetry (X, Z) including the longitudinal axis. Some moment of inertia are null $(I_{XZ}$ and $I_{YZ})$

- Writing in detail the equations of motion in the inertial frame, we obtain six differential equations, nonlinear because of the presence of trigonometric functions and the dependence of aeronautical forces of velocity and altitude through the air density.

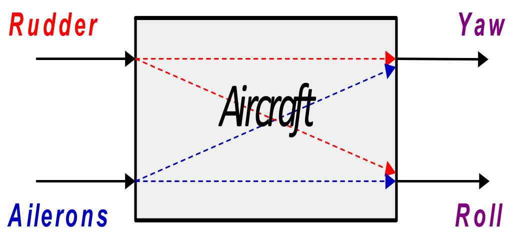
- These six equations can be decomposed into three equations describing the longitudinal motion and three other ones describing the lateral motion.

- Here we focus on the lateral motion. Additional assumptions are considered : the velocity is constant, the altitude is constant, the motion is rectilinear meaning that $\Theta = 0$ and we suppose that the deviations with respect to the equilibrium are small.

With these assumptions, the lateral motion is described by the following three linear time-invariant differential equations

$$\dot{\beta} = a_{11}\beta + a_{12}\dot{\Phi} - \dot{\Psi} + a_{14}\Phi + b_{11}\delta_r$$
$$\ddot{\Phi} = a_{21}\beta + a_{22}\dot{\Phi} + a_{23}\dot{\Psi} + b_{21}\delta_r + b_{22}\delta_a$$
$$\ddot{\Psi} = a_{31}\beta + a_{32}\dot{\Phi} + a_{33}\dot{\Psi} + b_{31}\delta_r + b_{32}\delta_a$$

with a_{ij} and $b_{ij} \in \mathbb{R}$ whose values depend of operating conditions and the physical parameters of aircraft. δ_r and δ_a are respectively the rudder and aileron angles (controls).



Choosing the state vector $x = \begin{bmatrix} \beta & \dot{\Phi} & \dot{\Psi} & \Phi \end{bmatrix}^T$, denoting the control $u = \begin{bmatrix} \delta_r & \delta_a \end{bmatrix}^T$ and defining the outputs $y = \begin{bmatrix} \dot{\Psi} & \Phi \end{bmatrix}$, the state-space model is given by

$$\left\{ \begin{array}{lcl} \dot{x} & = & \underbrace{\begin{bmatrix} a_{11} & a_{12} & -1 & a_{14} \\ a_{12} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ 0 & 0 \end{bmatrix}}_{B} u \\ y & = & \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} x \end{array} \right.$$

I.2) Analysis in Open Loop

The characteristic polynomial is given by

$$\det(sI - A) = (s^2 + 2\zeta_d \omega_d s + \omega_d^2)(s + 1/\tau_s)(s + 1/\tau_r) = 0$$

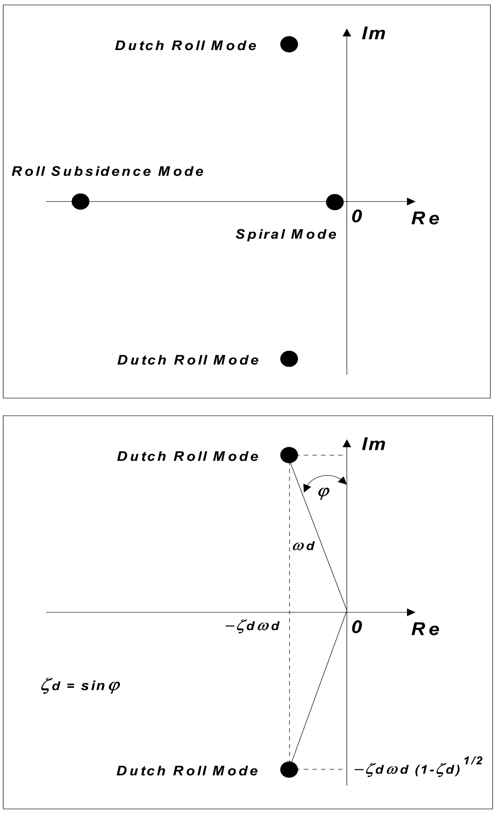
Three modes can be identified

1. The Dutch Roll mode *(mode Roulis Hollandais)*. This mode is due to a coupling between roll and yaw motion. This oscillatory mode is not sufficiently damped.
2. The Spiral mode *(mode Spiral)*. This stable mode is very slow (close to imaginary axis).
3. The Roll Subsidence mode *(mode Roulis Amorti)*. This aperiodic mode is well damped and no dominant.

I.2) Analysis in Open Loop

For a classical plane (Airbus A320), for a velocity of 0.8 Mach (~ 980 km/h), an altitude of 40000 feet (~ 12000 m) and a mass of 66 tonnes, the state-space model is

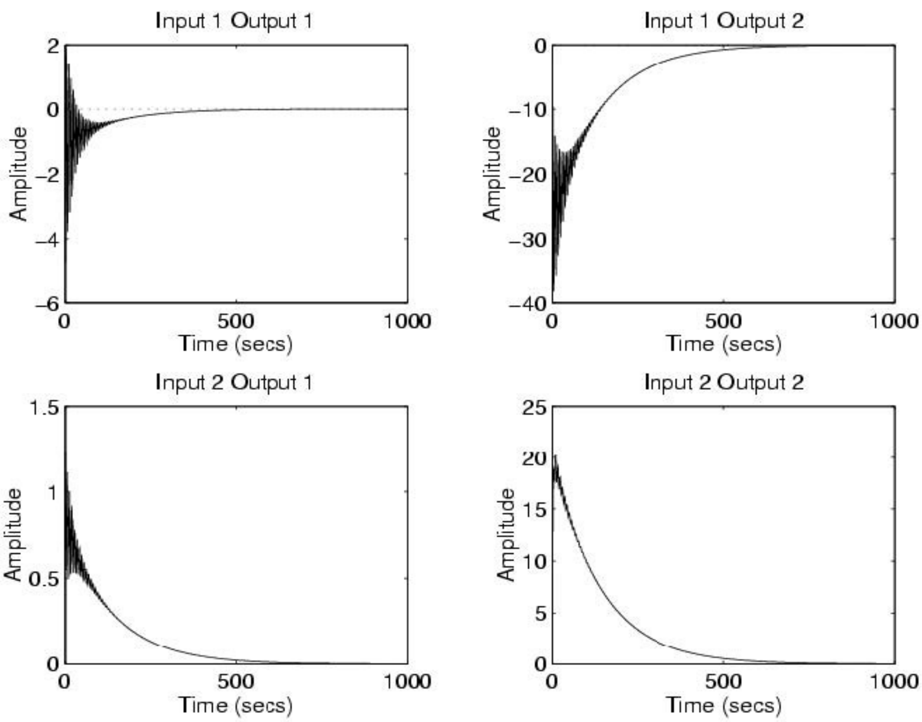
$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} -0.056 & 0.080 & -1 & 0.042 \\ -3.050 & -0.465 & 0.388 & 0 \\ 0.598 & -0.032 & -0.115 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0.073 & 0 \\ -4.750 & 1.230 \\ 1.530 & 10.630 \\ 0 & 0 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} x \end{cases}$$



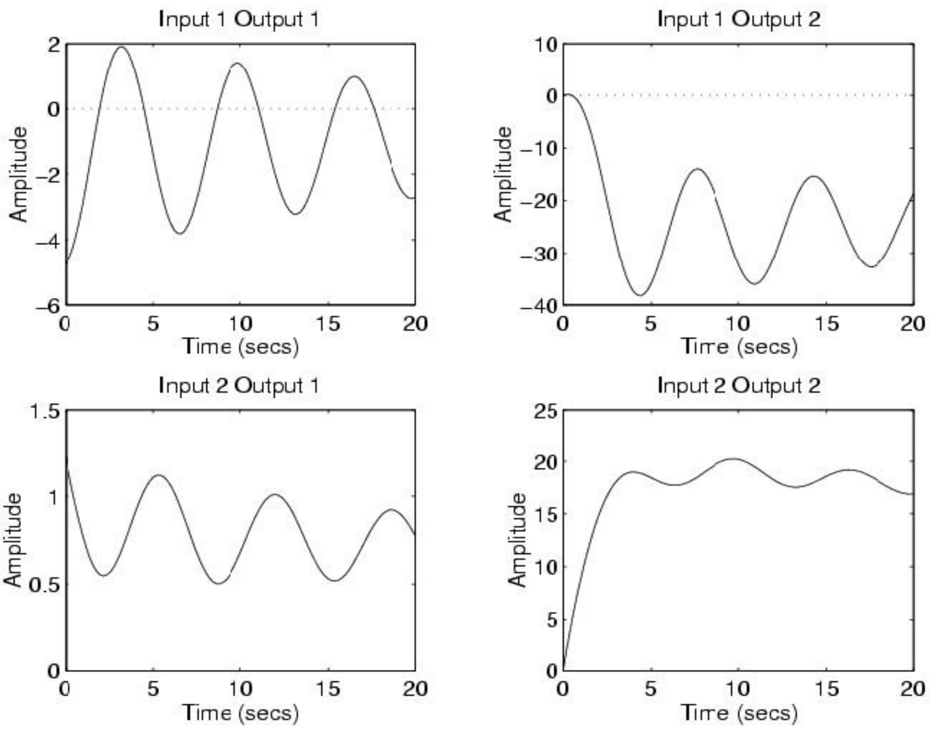
The modes are

1. Dutch Roll Mode: $-0.0308 \pm 0.9479i$, damping $\zeta_d = 0.0325$, natural frequency $\omega_d = 0.9484$ rd/s
2. Spiral Mode: -0.0098
3. Roll Subsidence Mode: -0.5646

Dutch Roll Mode



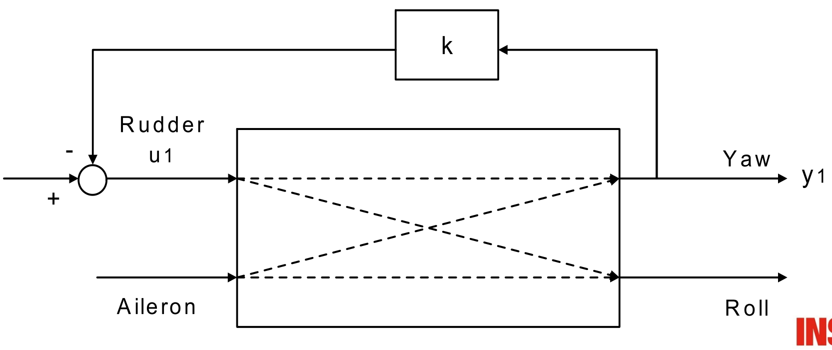
Impulse Response in open-loop

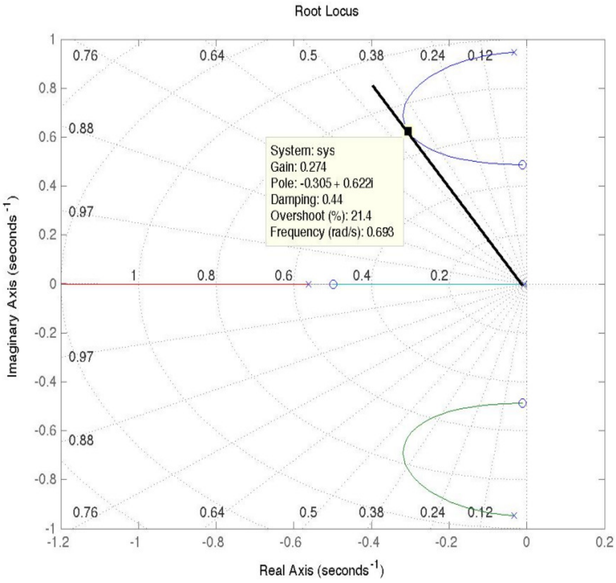


Impulse Response in open-loop

I.3) Analysis of a Monovariable Design

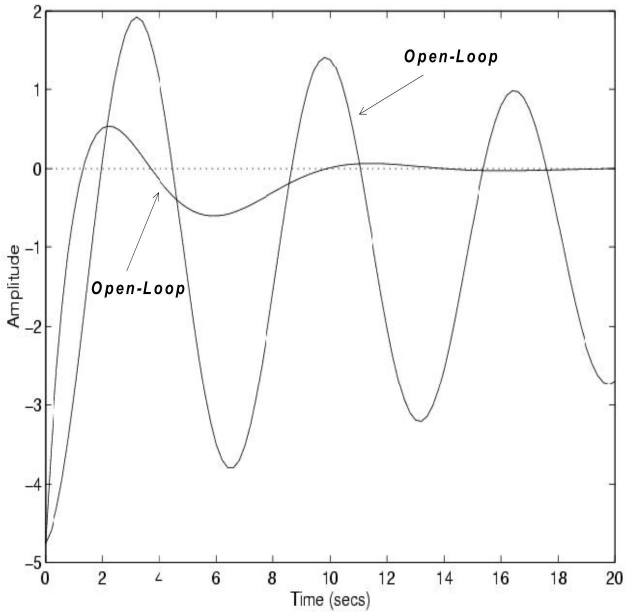
$$\left\{ \begin{array}{lcl} \dot{x} & = & \begin{bmatrix} -0.056 & 0.080 & -1 & 0.042 \\ -3.050 & -0.465 & 0.388 & 0 \\ 0.598 & -0.032 & -0.115 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \underbrace{\begin{bmatrix} 0.073 \\ -4.750 \\ 1.530 \\ 0 \end{bmatrix}}_{B_1} u_1 \\ y_1 & = & \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{C_1} x \end{array} \right.$$





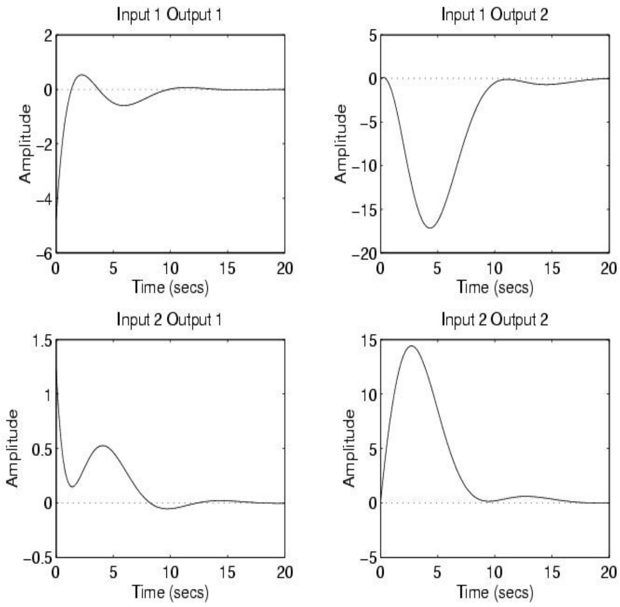
- It is possible to find a value of k improving the damping ratio of the Dutch roll mode
- The same value of k improves the spiral mode dynamic and the one of the roll subsidence mode
- the value of k is -0.274

Root-Locus $1 + kC_1(sI - A)^{-1}B_1 = 0, \quad k < 0$



As expected, the oscillations have been damped and the rise-time has been improved

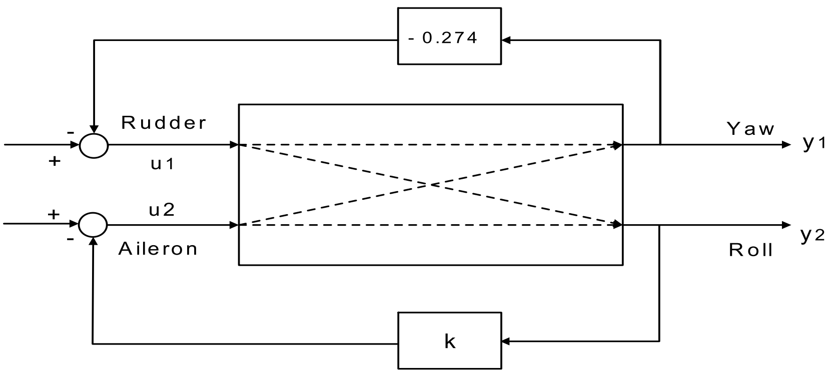
Open-Loop and Closed-Loop Impulse responses $u_1 \to y_1$

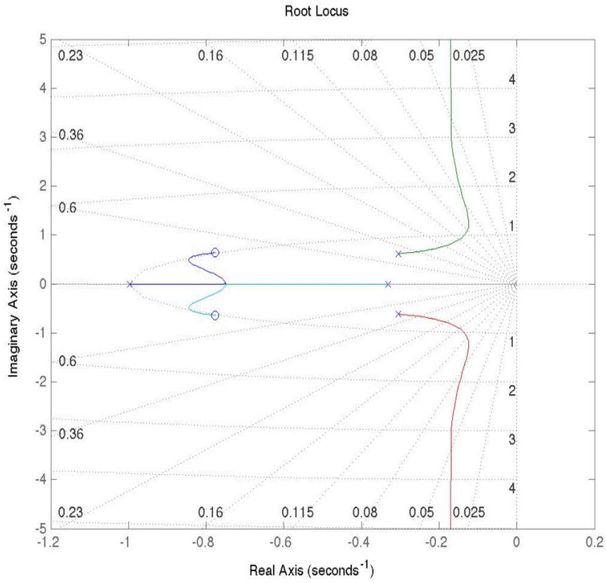


- There is an important overshoot for the impulse response $u_2 \rightarrow y_2$
- We can try to reduce the overshoot by closing the loop between y_2 and u_2

Open-Loop and Closed-Loop Impulse responses for the overall system

$$
\left\{
\begin{array}{lcl}
\dot{x} & = & \underbrace{\begin{bmatrix} -0.056 & 0.080 & -1 & 0.042 \\ -3.050 & -0.465 & 0.807 & 0 \\ 0.598 & -0.032 & -1.412 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{A+0.274.B_1.C_1} x + \underbrace{\begin{bmatrix} 0 \\ 1.230 \\ 10.630 \\ 0 \end{bmatrix}}_{B_2} u_2 \\
y_2 & = & \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}_{C_2} x
\end{array}
\right.
$$





- It is not possible to find a value of $k > 0$ improving the damping ratio of the oscillatory mode. A simple analysis shows that similarly, there does not exist a value of $k < 0$ improving the impulse response.
- Then it is not possible to reduce the overshoot of the impulse response $u_2 \to y_2$
- A different approach has to be used to solve appropriately the problem

Root-Locus $1 + kC_2(sI - A)^{-1}B_2 = 0, \;\; k > 0$

I.4) Outline of the Course

- Chapter I - Introductory Chapter
- Chapter II - State Feedback and Eigenstructure Assignment
- Chapter III - Eigenstructure Assignment by State Feedback : The Decoupling Problem
- Chapter IV - Controllability - Observability
- Chapter V - Models of Multivariable Systems : Their relations
- Chapter VI - State-Space Realization of Transfer Matrices

I.5) Some references

- P. Antzaklis and A.N. Michel. "Linear Systems". McGraw Hill, NewYork, 1997.
  (Good complete book dealing with Linear systems. Interesting but not for a first reading.)
- C. T. Chen. "Linear Systems Theory and Design". Holt, Rinehart and Wilson Inc., 1984.
  (Good complete book dealing with Linear systems.)
- A. Fossard. "Systèmes Multidimensionnels". Dunod. 1974.
  (*Book in french focused on the second part of the course. Does not contain a presentation of eigenstructure assignment. A. Fossard was Professor at SUPAERO.*

I.5) Some references (to go further)

- W. M. Wonham. "Linear Multivariable Control: a Geometric Approach". Springer Verlag, 1979.
  The geometric approach, not presented in this course, plays an important role in the theory of linear systems. The idea is to propose an intrinsic interpretation of the main concepts associated with linear systems (in terms of subspaces and linear algebra tools. Not treated in this couse)
- V. Kučera. "Discrete Linear Control : The Polynomial Approach". Chichester, Wiley. 1979.
  This book presents the approach based on the theory of polynomials. Some parts of the course uses this theory. This book goes beyond and proposes some design control methods in this context.
- B.D.O. Anderson and J.B. Moore."Optimal control: Linear Quadratic Methods". Prentice Hall, Englewood Cliffs, New Jersey. 1990.
  This book presents the optimal control methods design associated with a quadratic criterion. One of the best one in this domain.

MULTIVARIABLE SYSTEMS

Chapter II

State Feedback and
Eigenstructure Assignment

Objective of Chapter II

- Recall the specificities of a state feedback control law
- Compare the single-input case to the multi-input one
- Introduce several important definitions (left and right eigenvectors, eigenstructure)
- List the possibilities offered by an eigenstructure assignment control law
- Define the main steps for a design of such a control law

The seminal paper dealing with eigenstructure assignment :
- B.C Moore. " *On the flexibility offered by state-feedback in Multivariable Systems beyond closed-loop eigenvalue assignement*". IEEE Transactions on Automatic Control, pp. 689-692. October 1976.

Outline of Chapter II

II-1. State-Feedback Control

II-2. Right, Left-Eigenvectors - Eigenstructure

II-3. Properties of Eigenvectors

II-4. A Case Study: Sensitivity of Pole Assignment

II-5. Eigenstructure and Decoupling Control

II.6. An Illustrative Example

II.1) State-Feedback Control

We consider a system described in the state space by

$$\begin{cases} \dot{x} = Ax + Bu & A : n \times n \\ z = Cx & B : n \times m, \ C : p \times n \end{cases}$$

x being the state, u the input and z the controlled output.

Supposing x measurable, a state feedback control law can be considered as

$$u = -Kx + Hy_c, \quad K : m \times n, \quad H : m \times p$$

where y_c is the reference signal, K the state feedback gain and H a gain which can be used to adjust the static gain between y_c and z.

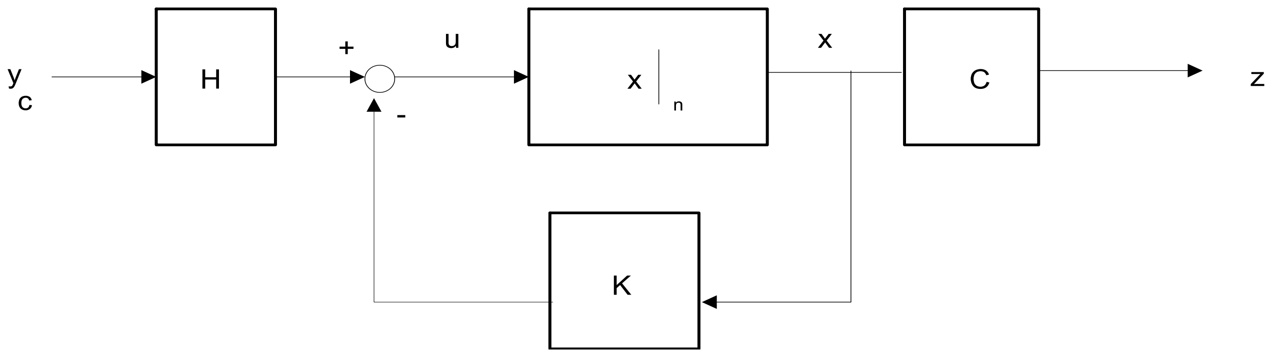


Fig II.1: State-Feedback Control

Closing the loop, we obtain

$$\begin{cases} \dot{x} = (A - BK)x + BHy_c \\ z = Cx \end{cases}$$

The matrix K offers $m \times n$ freedom degrees for acting on the closed-loop dynamical matrix.

- If the system is controllable, we can select arbitrarily all the eigenvalues of $A - BK$
- Only n freedom degrees are necessary in that case
- It remains $(m - 1) \times n$ freedom degrees which can be used
- They can be used for assigning other properties, for example, eigenvectors...

We an deduce that

- In the single-input case, if the system is controllable, there exists a unique matrix K assigning a given closed-loop eigenvalues set
- In the multivariable case, if the system is controllable, there are several matrices K assigning a given closed-loop eigenvalues set. For a given set, the number of ways for selecting the n parameters of matrix K is

$$N_K = \mathcal{C}_n^{n \times m} = \frac{(n \times m)!}{[(m-1) \times n]!\ n!}$$

For $m = 1$, $N_K = 1$

II.2) Right, Left-Eigenvectors - Eigenstructure

Let

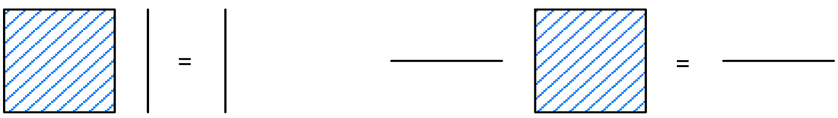
$$A_c = A - BK$$

The *right-eigenvector* v_i associated to the eigenvalue λ_i satisfies

$$A_c v_i = \lambda_i v_i, \quad i = 1, \cdots, n \qquad \text{(II.1)}$$

The *left-eigenvector* u_i associated to the eigenvalue λ_i satisfies

$$u_i A_c = \lambda_i u_i, \quad i = 1, \cdots, n \qquad \text{(II.2)}$$



(II.1)   (II.2)

u_i, v_i and λ_i, $i = 1, \cdots, n$ define the *eigenstructure* of the closed-loop system.

II.3) Properties of Eigenvectors

For the sequel, we suppose that the matrix A_c is diagonalizable. Multiplying (II.1) on the left by u_j leads to

$$\underbrace{u_j A_c}_{=\lambda_j u_j} v_i = u_j \overbrace{A_c v_i}^{=\lambda_i v_i} = \lambda_i u_j v_i = \lambda_j u_j v_i$$

and then

$$(\lambda_j - \lambda_i) u_j v_i = 0$$

We can deduce that

- If $\lambda_i \neq \lambda_j$ then $u_j v_i = 0$ meaning that vector u_j^T is orthogonal to vector v_i
- If $\lambda_i = \lambda_j$ then $u_j v_i = 0$ or $u_j v_i \neq 0$.
  Because the matrix of eigenvectors has to be invertible, we will have $u_j v_i \neq 0$ And without loss of generality, it is always possible to select $u_j v_i = 1$

Now defining respectively, the right and left eigenvectors matrices as

$$V = [v_1, v_2, \cdots, v_n] \quad \text{and} \quad U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

We have

$$U.V = I \text{ (Identity matrix)} \Rightarrow U = V^{-1}$$

$$U.A_c.V = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

II.4) A Case Study: Sensitivity of Pole Assignment

Consider the problem of eigenvalue assignment by state-feedback

$$\dot{x} = Ax + Bu \underset{(u=-Kx+Hy_c)}{\longrightarrow} \dot{x} = (A - BK)x + BHy_c$$

- The state-feedback control gain is designed using the model
- But the model only approximates the real system behavior

Problem

- How do the model errors impact the closed-loop eigenvalues assignment?
- What is the best state-feedback in terms of sensitivity of the closed-loop eigenvalues to model errors?

To evaluate the impact of error models on the closed-loop eigenvalues, we replace A and B by $A + \Delta A$ and $B + \Delta B$ where ΔA and ΔB represents the model errors

Due to errors model matrix A_c, eigenvectors v_i and eigenvalues λ_i must be respectively replaced by $A_c + \Delta A_c$, $v_i + \Delta v_i$ and $\lambda_i + \Delta\lambda_i$. Recall that

$$A_c v_i = \lambda_i v_i$$

We also have

$$(A_c + \Delta A_c)(v_i + \Delta v_i) = (\lambda_i + \Delta\lambda_i)(v_i + \Delta v_i)$$

$$A_c v_i + \Delta A_c v_i + A_c \Delta v_i + \Delta A_c \Delta v_i = \lambda_i v_i + \lambda_i \Delta v_i + \Delta\lambda_i v_i + \Delta\lambda_i \Delta v_i$$

Because we are interested by a sensitivity analysis, the second order terms can be neglected and then

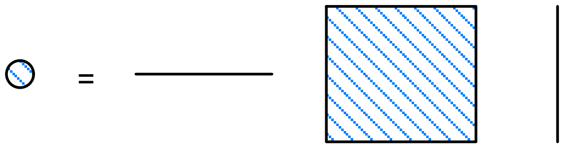
$$\underbrace{A_c v_i}_{=\lambda_i v_i} + \Delta A_c v_i + A_c \Delta v_i = \lambda_i v_i + \lambda_i \Delta v_i + \Delta\lambda_i v_i$$

Multiplying by the left-eigenvector u_i on the left leads to

$$u_i \Delta A_c v_i + \underbrace{u_i A_c}_{=\lambda_i u_i} \Delta v_i = \lambda_i u_i \Delta v_i + \Delta\lambda_i \underbrace{u_i v_i}_{=1}$$

and

$$\Delta\lambda_i = u_i \Delta A_c v_i$$



VECTOR NORM

Definition

Let E a vector space, a norm on E is an application f such that

$$\begin{aligned} f : E &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto f(x) \triangleq \|x\| \end{aligned}$$

and

i) $\forall x \in E, \quad f(x) = 0 \Rightarrow x = 0$

ii) $\forall (\lambda, x) \in \mathbb{R} \times E, \quad f(\lambda x) = |\lambda| .f(x)$ (Homogeneity property)

iii) $\forall (x, y) \in E^2, \quad f(x + y) \leqslant f(x) + f(y)$ (triangular Inequality)

MATRIX NORM

Definition

Let $\mathcal{M}_{n,m}(\mathbb{R})$ the set of $n \times m$ matrices defined over the field $\mathbb{R}$. A norm on $\mathcal{M}_{n,m}(\mathbb{R})$ is an application g such that

$$\begin{aligned} g : \mathcal{M}_{n,m}(\mathbb{R}) &\longrightarrow \mathbb{R}^+ \\ M &\longmapsto g(M) \triangleq |||M||| \end{aligned}$$

and

i) g is a norm on the vector space $\mathcal{M}_{n,m}(\mathbb{R})$

ii) $\forall (A,B) \in \mathcal{M}^2_{n,m}(\mathbb{R}), \quad g(A.B) \leqslant g(A).g(B)$ (multiplicative property)

Consider a matrix A. An important example of a matrix norm is

$$|||A||| = \sup_{v \neq 0} \underbrace{\frac{\|Av\|}{\|v\|}}_{\text{Euclidian norm}} = \lambda_{\max}(AA^T)$$

This norm satisfies the following property

$$\forall v \qquad \|Av\| \leqslant |||A|||\ \|v\|$$

Returning to our problem

$$\Delta\lambda_i = u_i \Delta A_c v_i \implies |\Delta\lambda_i| \leqslant |||\Delta A_c|||\ \underbrace{\|u_i\|\ \|v_i\|}_{C_i}$$

- To minimize $|\Delta\lambda_i|$, one way consists in minimizing C_i.

- It is possible to show that minimizing C_i can be translated into the minimization of the condition number C of the matrix of eigenvectors V defined as

$$C = |||V^{-1}|||\ |||V|||$$

- This method is implemented in the procedure "**place**" of Matlab.

place(A,B,[list of closed-loop eigenvalues]))

All the eigenvalues have to be chosen distinct

- For details, see: J. Kautsky and N.K. Nichols. "*Robust Pole assignment in systems to structured perturbations*". Systems and Control Letters, 15, p.373-380, 1990.

II.5) Eigenstructure and Decoupling Control

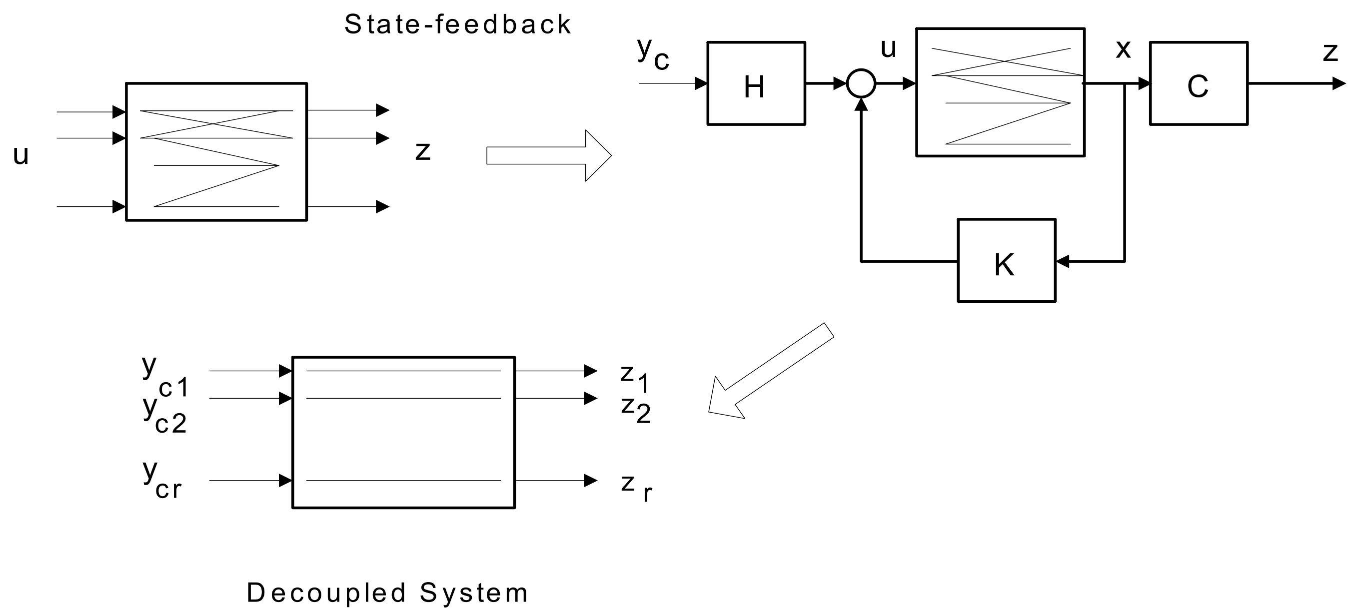
$$\dot{x} = Ax + Bu \underset{(u=-Kx+Hy_c)}{\longrightarrow} \dot{x} = A_c x + BHy_c$$

We can write the closed-loop model in the modal basis which corresponds to the change of coordinates

$$x = Vx_*$$

We obtain

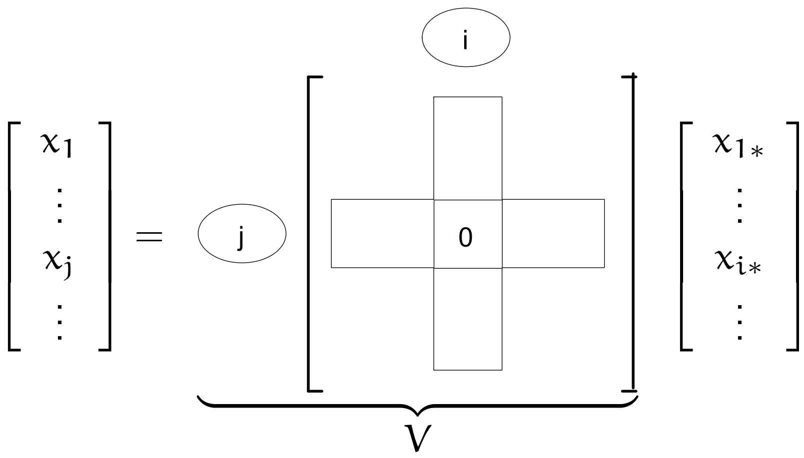
$$\begin{cases} \dot{x}_* & = \underbrace{\overbrace{U}^{=V^{-1}} A_c V}_{=\Lambda \text{ (diagonal)}} x_* + UBH\, y_c \\ z & = C\, x_* \end{cases}$$



To design a decoupling control law, we can enforce the structures of the matricial dependences between the signal vectors of interest.

Decoupling States/Modes

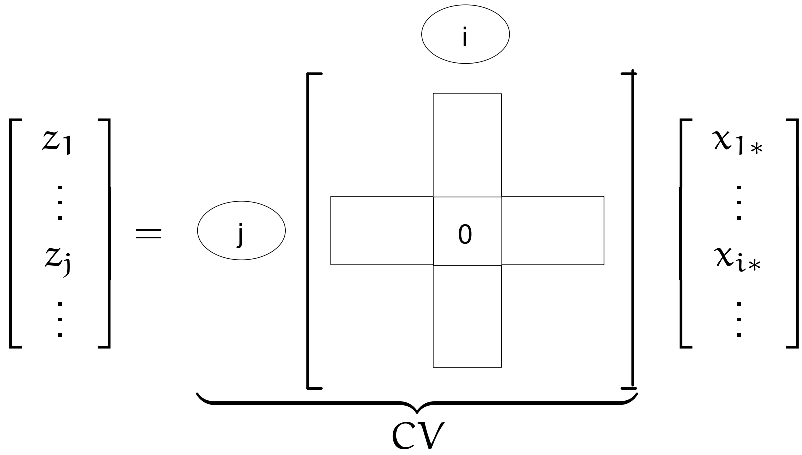
$$x = V\, x_*$$



The state x_j will not be inflenced by the mode x_{i*}

Decoupling Outputs/Modes

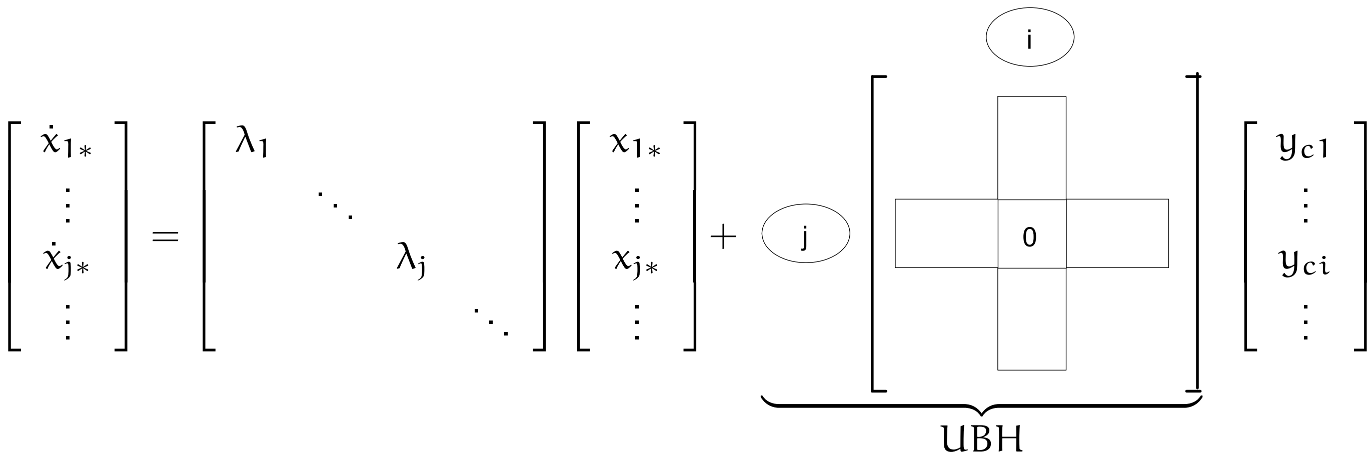
$$z = CV\, x_*$$



The output z_j will be independent of the mode x_{i*}

Decoupling References/Modes

$$\dot{x}_* = \Lambda\, x_* + UBH\, y_c$$



- The mode x_{j*} will be independent of the reference y_{ci}.

- The idea is to use the previous structural conditions to impose, when it is possible, a decoupling between references and outputs.

II.6) An Illustrative Example

$$\begin{cases} x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ z = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} x \end{cases}$$

- This system is unstable (Why?)

Problem: *Design a state-feedback which stabilizes the system and assigns the closed-loop eigenvalues* -1, -2 *and* -3 *while decoupling the references and the outputs in the following way*

$$y_c \begin{cases} y_{c_1} \longrightarrow \boxed{\lambda_1 = -1} \longrightarrow z_1 \\ y_{2_c} \longrightarrow \boxed{\lambda_2 = -2,\ \lambda_3 = -3} \longrightarrow z_2 \end{cases} \Bigg\} y$$

From the previous specifications, we can deduce that the closed-loop state model will be given by

$$\begin{cases} \dot{x}_* = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}}_{=\Lambda = UA_cV} x_* + \underbrace{\begin{bmatrix} \star & 0 \\ 0 & \star \\ 0 & \star \end{bmatrix}}_{=UBH} \begin{bmatrix} y_{c1} \\ y_{c2} \end{bmatrix} \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \star & 0 & 0 \\ 0 & \star & \star \end{bmatrix}}_{=CV} x_* \end{cases}$$

where $\star$ stands for a free parameter.

- We begin by the outputs to deduce the constraints imposed on the right eigenvectors. We have

$$CV = C\left[\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}\right] = \left[\begin{array}{ccc} \star & 0 & 0 \\ 0 & \star & \star \end{array}\right]$$

$$Cv_1 = \left[\begin{array}{c} \star \\ 0 \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 1 \end{array}\right]\left[\begin{array}{c} v_{11} \\ v_{12} \\ v_{13} \end{array}\right] = \left[\begin{array}{c} v_{11} \\ -v_{11} + v_{12} + v_{13} \end{array}\right]$$

Then $v_{11} = \star$ (free) and $v_{12} + v_{13} = v_{11}$.

$$Cv_2 = \left[\begin{array}{c} v_{21} \\ -v_{21} + v_{22} + v_{23} \end{array}\right] = \left[\begin{array}{c} 0 \\ \star \end{array}\right] \Longrightarrow \left\{\begin{array}{c} v_{21} = 0 \\ v_{22} + v_{23} = \star \end{array}\right.$$

$$Cv_3 = \left[\begin{array}{c} v_{31} \\ -v_{31} + v_{32} + v_{33} \end{array}\right] = \left[\begin{array}{c} 0 \\ \star \end{array}\right] \Longrightarrow \left\{\begin{array}{c} v_{31} = 0 \\ v_{32} + v_{33} = \star \end{array}\right.$$

At this stage, we deduce the constraints the right eigenvectors have to satisfy

$$V = \left[\begin{array}{ccc} a & 0 & 0 \\ b & c & e \\ a-b & d & f \end{array}\right] \text{ must be invertible}$$

From V, it is possible to compute $U = V^{-1}$. Then we can deduce the contraints H has to satisfy

$$UBH = UB \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} = \begin{bmatrix} \star & 0 \\ 0 & \star \\ 0 & \star \end{bmatrix}$$

- The controllability f the pair (A, B) ensures that the closed-loop eigenvalues can be chosen arbitrarily

- Now: Is it possible to satisfy the contraints imposed by the decoupling problem? (See the next chapter)

MULTIVARIABLE SYSTEMS

Chapter III

Eigenstructure Assignment by State-Feedback: The decoupling Problem

Objective of Chapter III

Outline of Chapter III

III-1. Eigenvector Subspaces - Degrees of freedom

III-2. Algorithm for the Design of K

III-3. Computation of $N(\lambda)$ and $M(\lambda)$

III-4. Complex Closed-Loop Eigenvalues

III-5. Illustrative Example (continued)

III-6. Determination of the best right-eigenvectors

III.7. Extension to Static-Output Feedback

III.8. Illustrative Examples

III.1) Eigenvector Subspaces -Degrees of freedom

PROBLEM STATEMENT

- Consider the system

$$x = Ax + Bu$$

- We want to design a control law $u = -Kx + Hy_c$ such that

- The eigenvalues of $A - BK$ are located at desired values in the complex plane
- The eigenvectors of $A - BK$ can be selected to satisfy some specifications like, for example, decoupling constraints

- To answer the previous problem, we must determine the set of admissible eigenvectors v_i associated to the eigenvalue λ_i.

- A simple analysis shows that we have $n \times m$ freedom degrees associated to matrix K. n of them are used for assigning the eigenvalues λ_i, $i = 1, \cdots, n$. It remains $(n-1) \times m$ freedom degrees which can be used to select appropriate eigenvectors.

If λ_i is an eigenvalue of A_c and v_i the associated eigenvector, we can write

$$(A - BK)v_i = \lambda_i v_i \Longrightarrow (\lambda_i I - A)v_i + BKv_i = 0$$

which can also be written

$$\begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \begin{bmatrix} v_i \\ Kv_i \end{bmatrix} = 0$$

If $w_i \triangleq Kv_i$, Then

$$\underbrace{\begin{bmatrix} \lambda_i I - A & B \end{bmatrix}}_{Q(\lambda_i)} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0$$

We can deduce that the vector $\begin{bmatrix} v_i \\ w_i \end{bmatrix}$ is in the null space of the matrix $Q(\lambda_i)$ whose dimensions are

$$Q(\lambda_i) = \underbrace{\begin{bmatrix} \lambda_i I - A & B \end{bmatrix}}_{n+m} | n$$

On one hand, it is well known that if the system is controllable (assumption which will be adopted), we have by the *Popov-Belevitch-Hautus* test that

$$\text{Rank}\,[Q(\lambda)] = n \quad \forall \lambda \in \mathbb{C}$$

On the other one, from the Rank-Nullity Theorem of linear algebra, we can write

$$\dim\,[\text{Ker}(Q(\lambda))] + \dim\,[\text{Im}(Q(\lambda))] = n + m$$

where "Ker" denotes the kernel or null space and "Im" the image or the range. Then

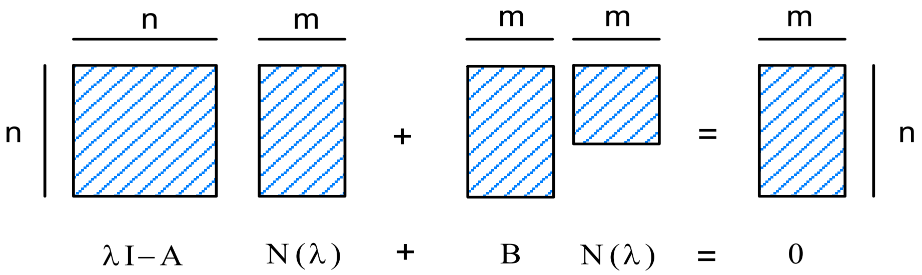
$$\dim\,[\text{Ker}(Q(\lambda))] = n + m - \underbrace{\dim\,[\text{Im}(Q(\lambda))]}_{=\text{Rank}[Q(\lambda)]=n} = m$$

Let $R(\lambda)$ be a basis of $\text{Ker}(Q(\lambda)$, it can be written

$$R(\lambda) = \overbrace{\begin{bmatrix} N(\lambda) \\ M(\lambda) \end{bmatrix}}^{m} \begin{matrix} | \, n \\ | \, m \end{matrix} \quad \text{and } Q(\lambda)R(\lambda) = 0$$

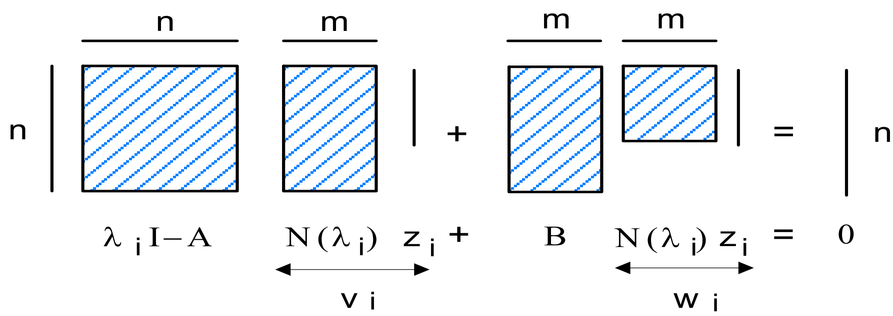
Developing $Q(\lambda)R(\lambda)$, we obtain

$$(\lambda I - A)\,N(\lambda) + BM(\lambda) = 0$$



For $\lambda = \lambda_i$,we can choose $z_i \in \mathbb{R}^m$ such that

$$(\lambda_i I - A)\,\underbrace{N(\lambda_i)z_i}_{=v_i} + \underbrace{BM(\lambda_i)z_i}_{=w_i} = 0$$



and

$$\begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0 \Longrightarrow v_i = N(\lambda_i)z_i \text{ and } w_i = Kv_i = M(\lambda_i)z_i,\ i = 1, \cdots, n$$

For a given λ_i, the degree of freedoms are contained in z_i. Defining now

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \text{ and } W = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$$

We have

$$KV = W \Rightarrow K = WV^{-1}$$

In term of subspaces, for a given λ_i, the right eigenvector satisfies

$$v_i = N(\lambda_i)z_i \in \text{Im}(N(\lambda_i))$$

III.2) Algorithm for the Design of K

The algorithm follows from the previous developments.

1. Choose the n closed-loop eigenvalues λ_i, $i = 1, \cdots, n$
2. For each λ_i, determination of $N(\lambda_i)$ and $M(\lambda_i)$
3. Selection of right eigenvectors v_i satisfying the contraints

   $$v_i = N(\lambda_i)z_i, \quad i = 1, \cdots, n$$

   If such a z_i does not exist, the problem has not a strict solution. It is possible to deduce an approximate solution (see next paragraph)
4. Compute

   $$w_i = M(\lambda_i)z_i \quad i = 1, \cdots, n$$

5. From $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$, the expression for K is

   $$K = WV^{-1}$$

III.3) Computation of $N(\lambda)$ and $M(\lambda)$

Numerical Approach (Here with MATLAB)

```
% Enter matrices A and B
        A=[...]; B=[...];
% Extract dimensions n and m
        [n,m]=size(B);
% Determination of the null space of Q(λ)
        Qlambda=[lambda*eye(n)-A   B];
        Rlambda=null(Qlambda);
% Determination of N(λ) and M(λ)
        Nlambda=Rlambda(1:n,:);
        Mlambda=Rlambda(n+1:n+m,:);
```


Analytical Approach

It is easy to see that

$$N(\lambda) = -(\lambda I - A)^{-1}B \text{ and } M(\lambda) = I$$

satisfies

$$(\lambda I - A)N(\lambda) + BM(\lambda) = 0$$

With this solution, it is not possible to assign a closed-loop eigenvalue equal to an eigenvalue of A (open-loop eigenvalue) because in that case $\lambda I - A$ is not invertible.

III.4) Complex Closed-Loop Eigenvalues

- The case where the set of closed-loop eigenvalues contains complex eigenvalues has to be examined with some care.

- Without loss of generality, consider that in the set, there is one complex eigenvalue, say λ_1.

- In that case, it is well known that the complex conjugate of λ_1 has to be included in this set, say $\lambda_1 = \overline{\lambda_2}$.

The associated eigenvectors satisfy $v_1 = \overline{v_2}$. To prove this fact, recall that

$$A_c v_1 = \lambda_1 v_1 \text{ and } A_c v_2 = \lambda_2 v_2$$

Taking the conjugate

$$\overline{A_c}\overline{v_1} = \overline{\lambda_1}\overline{v_1} = \lambda_2 \overline{v_1} = A_c \overline{v_1}$$

because A_c is a real matrix and $\overline{v_1}$ is the eigenvector associated with $\lambda_2 \Rightarrow v_2 = \overline{v_1}$.

Moreover

$$v_1 = N(\lambda_1) z_1 \Longrightarrow v_2 = \overline{v_1} = N(\lambda_2) z_2 = \overline{N(\lambda_1)}\overline{z_1} = N(\lambda_2)\overline{z_1}$$
$$\Longrightarrow z_2 = \overline{z_1} \Longrightarrow w_2 = \overline{w_1}$$

Writing now $v_1 = v_{R_1} + j v_{I_1}$ and $w_1 = w_{R_1} + j w_{I_1}$, we have

$$V = \begin{bmatrix} v_{R_1} + j v_{I_1} & v_{R_1} - j v_{I_1} & v_3 & \cdots & v_n \end{bmatrix}$$
$$W = \begin{bmatrix} w_{R_1} + j w_{I_1} & w_{R_1} - j w_{I_1} & w_3 & \cdots & w_n \end{bmatrix}$$

The gain K is given by $KV = W$

$$K \begin{bmatrix} v_{R_1} + j v_{I_1} & v_{R_1} - j v_{I_1} & v_3 & \cdots & v_n \end{bmatrix} =$$
$$\begin{bmatrix} w_{R_1} + j w_{I_1} & w_{R_1} - j w_{I_1} & w_3 & \cdots & w_n \end{bmatrix}$$

Multiplying on the right by the matrix

$$\begin{bmatrix} \frac{1}{2} & -j\frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & j\frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

leads to

$$K \begin{bmatrix} v_{R_1} & v_{I_1} & v_3 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} w_{R_1} & w_{I_1} & w_3 & \cdots & w_n \end{bmatrix}$$

A generalization to the case where several complex eigenvalues belong to the closed-loop eigenvalues set.

III.5) Illustrative Example (continued)

We recall the system under study

$$\begin{cases} x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ z = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} x \end{cases}$$

Problem: *Design a state-feedback which stabilizes the system and assigns the closed-loop eigenvalues* -1, -2 *and* -3 *while decoupling the references and the outputs in the following way*

$$y_c \begin{cases} y_{c_1} \longrightarrow \boxed{\lambda_1 = -1} \longrightarrow y_1 \\ y_{2_c} \longrightarrow \boxed{\lambda_2 = -2,\ \lambda_3 = -3} \longrightarrow y_2 \end{cases} y$$

From the previous specifications, we deduced that the closed-loop state model will be given by

$$\begin{cases} \dot{x}_* = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}}_{=\Lambda=UA_cV} x_* + \underbrace{\begin{bmatrix} \bigstar & 0 \\ 0 & \bigstar \\ 0 & \bigstar \end{bmatrix}}_{=UBH} \begin{bmatrix} y_{c1} \\ y_{c2} \end{bmatrix} \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \bigstar & 0 & 0 \\ 0 & \bigstar & \bigstar \end{bmatrix}}_{=CV} x_* \end{cases}$$

where $\bigstar$ stands for a free parameter.

The constraints on the right eigenvectors given by

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

can be summarized as

- $v_{11} = \bigstar$ and $v_{12} + v_{13} = v_{11}$
- $v_{21} = 0$ and $v_{22} + v_{23} = \bigstar$
- $v_{31} = 0$ and $v_{32} + v_{33} = \bigstar$

Having in mind the previous contraints, we deduced the constraints the right eigenvectors have to satisfy

$$V = \begin{bmatrix} a & 0 & 0 \\ b & c & e \\ a-b & d & f \end{bmatrix} \text{ must be invertible}$$

Determination of $N(\lambda)$ and $M(\lambda)$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix} \qquad (\lambda I - A)^{-1} = \frac{1}{\lambda^3}\begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$$

$$N(\lambda) = -(\lambda I - A)^{-1} B = \begin{bmatrix} -\lambda^{-1} & 0 \\ -\lambda^{-1} & -\lambda^{-2} \\ 0 & -\lambda^{-1} \end{bmatrix} \text{ and } M(\lambda) = I$$

Determination of right-eigenvectors

- $v_1 = N(-1)z_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} a \\ b \\ a-b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ with

$z_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$

- $v_2 = N(-2)z_2 = \begin{bmatrix} 1/2 & 0 \\ 1/2 & -1/4 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ with

$z_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix} = w_2$

- $v_3 = N(-3)z_3 = \begin{bmatrix} 1/3 & 0 \\ 1/3 & -1/9 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} z_{31} \\ z_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ with

$z_2 = \begin{bmatrix} 0 \\ -9 \end{bmatrix} = w_3$

Determination of K

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -9 \end{bmatrix}$$

and then

$$U = V^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 1 \\ 2 & -2 & -1 \end{bmatrix}, \quad K = WV^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 6 & 5 \end{bmatrix}$$

Determination of H

$$UBH = \begin{bmatrix} \star & 0 \\ 0 & \star \\ 0 & \star \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 1 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{UB} \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}_{H}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \Longrightarrow H = I$$

III.6) Determination of the Best Right-Eigenvectors

Case where strict satisfaction of constraints is impossible

- In some cases, it is impossible to exactly satisfy the contraints on a right-eigenvector

- The idea is to find an approximation in a best sense. Suppose that the desired right eigenvector is v_i^d associated with the eigenvalue λ_i. The possible right-eigenvectors associated with λ_i are expressed as

$$N(\lambda_i)z_i$$

We can defined the error

$$\varepsilon(z_i) = v_i^d - N(\lambda_i)z_i$$

and the criterion

$$J(z_i) = \varepsilon^T(z_i)\varepsilon(z_i) = \|\varepsilon(z_i)\|^2$$

The best right-eigenvector in the sense of the least-squares corresponds to the value z_i obtaining by solving the optimization problem

$$\min_{z_i} J(z_i)$$

The stationnary condition leads to

$$\left.\frac{dJ(z_i)}{dz_i}\right|_{\hat{z}_i} = 0 = -N(\lambda_i)^T(v_i^d - N(\lambda_i)\hat{z}_i)$$

The solution is given by

$$\hat{z}_i = \left[N(\lambda_i)^T N(\lambda_i)\right]^{-1} N(\lambda_i)^T v_i^d$$

This corresponds to a minimum because

$$\left.\frac{d^2J(z_i)}{dz_i^2}\right|_{\hat{z}_i} = N(\lambda_i)^T N(\lambda_i) \geqslant 0$$

Case where only some elements of right-eigenvectors are fixed

For example, the desired eigenvector is

$$v_i^d = \begin{bmatrix} 0 \\ \star \\ 1 \\ \star \\ \vdots \end{bmatrix} \text{ with } N(\lambda_i) = \begin{bmatrix} ========== \\ \star\star\star\star\star\star \\ ========== \\ \star\star\star\star\star\star \\ \vdots \end{bmatrix}$$

We build a vector only composed of the specified elements

$$\widetilde{v_i^d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ with } \widetilde{N(\lambda_i)} = \begin{bmatrix} ========== \\ ========== \end{bmatrix}$$

The right-eigenvector is obtained by

$$z_i = \left[\widetilde{N(\lambda_i)}^T \widetilde{N(\lambda_i)}\right]^{-1} \widetilde{N(\lambda_i)}\widetilde{v_i^d}$$

III.7) Extension to Static-Output Feedback

Consider the system described by

$$\dot{x} = Ax + Bu$$
$$y = z = Cx$$

- We suppose that only an output is measurable.

- In addition, we suppose that the measurable output is the controlled output.

The considered control law is now given by

$$u = -Ly + Hy_c = -LCx + Hy_c$$

The closed-loop system becomes

$$\dot{x} = (A - BLC)x + BHy_c$$
$$z = Cx$$

- The main difference is that the number of degrees is of freedom is $m \times p$ while it was $m \times n$ for state-feedback. Usually $p < n$. Recall that

$$v_i = N(\lambda_i)z_i \quad z_i \in \mathbb{R}^m$$

- Only p eigenvalues and p eigenvectors can be fixed, the remaining $n - p$ eigenvalues and eigenvectors will be assigned automatically and it is necessary to verify that this assignment is compatible with the practical constraints

1. Choose the p closed-loop eigenvalues $\lambda_i, \; i = 1, \cdots, p$
2. For each λ_i, determination of $N(\lambda_i)$ and $M(\lambda_i)$
3. Selection of right eigenvectors v_i satisfying the contraints
$$v_i = N(\lambda_i) z_i, \quad i = 1, \cdots, p$$
such that $C[v_1 \ldots v_p]$ is invertible
4. Compute
$$w_i = M(\lambda_i) z_i, \quad i = 1, \cdots, p$$
5. Compute
$$L = W[CV]^{-1} = [w_1 \ldots w_p][Cv_1 \ldots Cv_p]^{-1}$$
6. Verification that the remaining $n-p$ eigenvalues are located in a satisfactory way

III.8) Illustrative Examples

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

Eigenvalues: -1 and -2 with eigenvectors $v_{d_1} = \begin{bmatrix} \star \\ 0 \\ 1 \end{bmatrix}$ and $v_{d_2} = \begin{bmatrix} 0 \\ 1 \\ \star \end{bmatrix}$

$$N(\lambda) = -(\lambda I - A)^{-1} B = \begin{bmatrix} -\lambda^{-1} & 0 \\ 0 & -(\lambda - 1)^{-1} \\ -(\lambda + 5)^{-1} & 0 \end{bmatrix} \qquad M(\lambda) = I$$

$$v_{d_1} = N(-1)z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ -1/4 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$$v_{d_2} = N(-2)z_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ -1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$L = \underbrace{\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}}_{W} \left(\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{CV} \right)^{-1} = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A - BLC = \begin{bmatrix} -4 & 0 & -12 \\ 0 & -2 & 0 \\ -4 & 0 & -17 \end{bmatrix} \text{ and } \det(sI - A + BLC) = (s+1)(s+2)(s+20)$$

The last eigenvalue is non dominant when compared to the two fixed ones. For this example, the static-output feedback control is satisfactory.

Now if the specifications are

Eigenvalues: -1 and -2 with eigenvectors $v_{d_1} = \begin{bmatrix} 0 \\ 1 \\ \star \end{bmatrix}$ and $v_{d_2} = \begin{bmatrix} \star \\ 0 \\ 1 \end{bmatrix}$

$$v_{d_1} = N(-1)z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ -1/4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_{d_2} = N(-2)z_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ -1/3 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}$$

$$L = \underbrace{\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}}_{W} \Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -3/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{CV} \Bigg)^{-1} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/2 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A - BLC = \begin{bmatrix} 2 & 0 & 6 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } \det(sI - A + BLC) = (s+1)(s+2)(s-5)$$

In that case, the last eigenvalue is unstable. From a practical point of view, the control is not admissible.

MULTIVARIABLE SYSTEMS

Chapter IV

Controllability - Observability

Objective of Chapter IV

- Define the important concepts of controllability and observability
- Understand the implication of controllability and observability and their relations with the different models
- Develop the tests of controllability and observability in the multivariable case

Outline of Chapter IV

IV-1. Controllability

IV-2. Test for Diagonal State-Space Model

IV-3. Observability

IV-4. Test for Diagonal State-Space Model

IV-5. Example

IV-6. Other criteria

IV-7. Duality

IV.8. Models and Structures

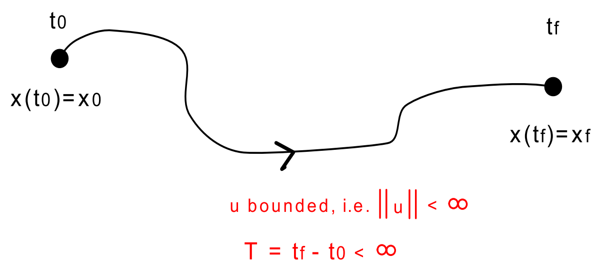
IV.1) Controllability

Consider a system described by

$$\dot{x} = Ax + Bu, \qquad \dim\ x = n, \quad \dim\ u = m \tag{1}$$

Definition

System (1) is controllable if for all initial state x_0 and all terninal state x_f, there exists a bounded control u driving the system from x_0 to x_f in a finite time T.



In the state-space, the system trajectory is expressed by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

The problem consists in determining u and T which satisfies

$$\int_{t_0}^{T+t_0} e^{A(T+t_0-\tau)}Bu(\tau)d\tau = \left[x_f - e^{AT}x_0\right]$$

KALMAN TEST

Theorem

System (1) is controllable if and only if

$$\text{Rank}\,[Q_c] = \dim\ x = n$$

where Q_c is the Kalman controllability matrix defined by

$$Q_C = \left[\begin{array}{cccccc} B & AB & A^2B & \cdots & A^{n-1}B \end{array}\right]$$

Q_C is a rectangular matrix of order $n \times (n.m)$

$$Q_C = \left[\underbrace{B}_{m}\ \underbrace{AB}_{m}\ \underbrace{A^2B}_{m}\ \cdots\ \underbrace{A^{n-1}B}_{m}\right] \Big| n \to \text{Rank}\,[Q_C] = m \Longleftrightarrow \det\left[Q_C Q_C^T \neq 0\right]$$

SOME REMARKS

- Kalman test gives a binary response. When the system is not controllable, it does not explain why!!

- The determinant is not a good measure for the rank and then for singularity of a matrix. Consider the matrix

$$M = \left[\begin{array}{cc} 10 & 10/\epsilon \\ 0 & 10 \end{array}\right]$$

det $[M] = 100$ independent of ε. But if ε is small, the two lines of the matrix M are more and more colinear leading to a singular matrix while the determinant is clearly far to be null.

A good measure for singularity or for rank fullness is the minimum singular value or more effective, the condition number defines as

$$C(M) = \frac{\sigma_{max}(M)}{\sigma_{min}(M)} = \frac{\lambda_{max}^{1/2}\left[M^T M\right]}{\lambda_{min}^{1/2}\left[M^T M\right]}$$

where σ_{max} and σ_{min} are respectively the maximum and minimum singular value. For the matrix M, the singular values are given by

$$\sigma_{max} = \sqrt{100 + \frac{50}{\varepsilon^2} + \frac{50\sqrt{1+4\varepsilon^2}}{\varepsilon^2}} \qquad \sigma_{min} = \sqrt{100 + \frac{50}{\varepsilon^2} - \frac{50\sqrt{1+4\varepsilon^2}}{\varepsilon^2}}$$

For $\varepsilon = 0.1$ $\sigma_{max} = 100.99$ $\sigma_{min} = 0.99$ $C(M) = 102.01$

For $\varepsilon = 0.01$ $\sigma_{max} = 1000.09$ $\sigma_{min} = 0.09$ $C(M) = 11112.11$

Controllability is a structural property

This means that, for a given system, the property of controllability is independent of the considered state-space model.

$$\dot{x} = Ax + Bu \underset{x=Mx_*}{\longrightarrow} \dot{x}_* = M^{-1}AM\,x_* + M^{-1}Bu$$

$$\left\{ \begin{array}{c} \text{Controllable} \\ Q_0 = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right] \\ \det\left[Q_0 Q_0^T\right] \neq 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} Q_C = \left[\begin{array}{ccc} M^{-1}B & M^{-1}A\underbrace{MM^{-1}}_{I}B & \cdots \end{array}\right] \\ = M^{-1}\left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right] \\ = M^{-1}Q_0 \end{array} \right.$$

$$\det\left[Q_C Q_C^T\right] = \det\left[M^{-1}Q_0 Q_0^T (M^{-1})^T\right] = \det^2\left[M^{-1}\right]\det\left[Q_0 Q_0^T\right] \neq 0$$

IV.2) Test for Diagonal State-Space Model

Controllability Criterion for a Single-Input System (Diagonalizable)

In that case, there exists a state-vector such that the state model writes

$$x = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \qquad \lambda_i \neq \lambda_j \text{ for } i \neq j \qquad (2)$$

The Kalman matrix is given by

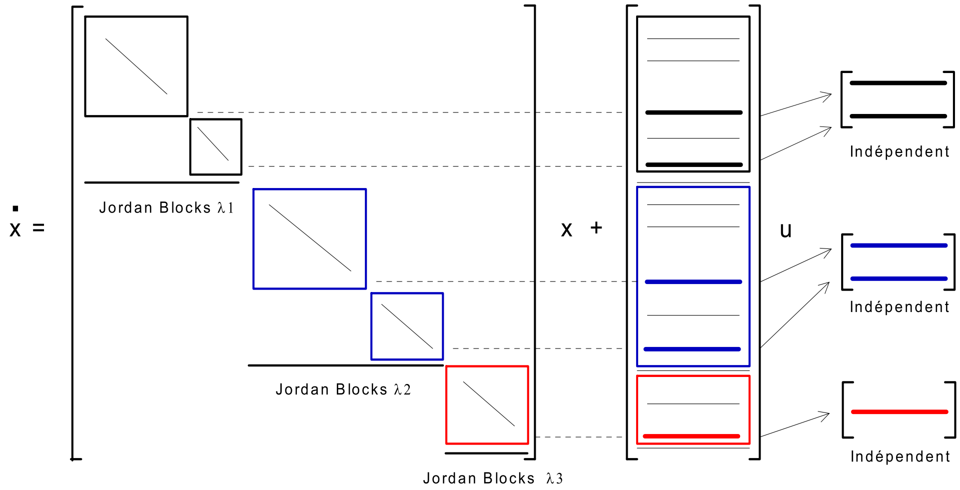
$$Q_C = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \cdots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 & \cdots & \lambda_2^{n-1} b_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & \cdots & \lambda_n^{n-1} b_n \end{bmatrix}$$

$$Q_C = \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{bmatrix} \underbrace{\begin{bmatrix} b_1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ b_2 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_n & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{\text{Vandermonde Matrix invertible if } \lambda_i \neq \lambda_j \text{ for } i \neq j}$$

Theorem

System (2) is controllable if and only if all the $b_i \neq 0, \quad i = 1, \cdots, n.$

Generalization: Controllability Criterion for a Multi-Inputs System



Theorem

System in a canonical Jordan Form is controllable if and only if the lines of the matrix B*, corresponding to the last lines of the Jordan blocks associated with a the same eigenvalue, are linearly independent.*

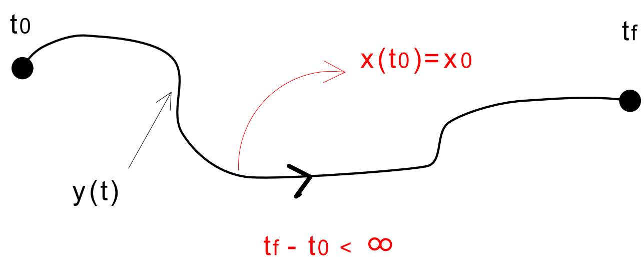
IV.3) Observability

Consider a system described by

$$\begin{cases} \dot{x} &= Ax + Bu, \quad \dim\ x = n, \quad \dim\ u = m \\ y &= Cx \quad \dim\ y = p \end{cases} \tag{2}$$

Definition

System (2) is observable if and only if from the knowledge of the output y on a finite interval of time $[t_0, t_f]$, it is possible to compute the initial state $x(t_0) = x_0$.



We have

$$
\begin{aligned}
y(t_0) &= Cx(t_0) + Du(t_0) \\
\dot{y}(t_0) &= CAx(t_0) + CBu(t_0) + D\dot{u}(t_0) \\
\ddot{y}(t_0) &= CA^2x(t_0) + CABu(t_0) + CB\ddot{u}(t_0) \\
&\vdots \\
y^{(n-1)}(t_0) &= CA^{n-1}x(t_0) + \sum_{j=0}^{n-2} CA^jBu^{(n-2-j)}(t_0) + Du^{(n-1)}(t_0)
\end{aligned}
$$

We can deduce

$$
\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t_0) = \begin{bmatrix} y(t_0) \\ \dot{y}(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{bmatrix} - \begin{bmatrix} Du(t_0) \\ CBu(t_0) + D\dot{u}(t_0) \\ \vdots \\ \sum_{j=0}^{n-2} CA^jBu^{(n-2-j)}(t_0) + Du^{(n-1)}(t_0) \end{bmatrix}
$$

The system is observable if and only if (Kalman test)

$$\text{Rank}\,[Q_O] = \dim \ x = n$$

where the Kalman observability matrix Q_O is defined as

$$Q_O = \left[\begin{array}{c} C \\ CA \\ \vdots \\ CA^{n-1} \end{array}\right] \begin{array}{l} |\ p \\ |\ p \\ \\ |\ p \end{array}$$

with the columns of Q_O spanning n.

Equivalently, the system is observable if and only if $\det\left[Q_O^T Q_O\right] \neq 0$.

SOME REMARKS

- Kalman test gives a binary response. When the system is not observable, it does not explain why!!
- The determinant is not a good measure for the rank and then for singularity of a matrix.
- For linear systems, the observability does not depend of the input. In the nonlinear case, it depends of the input.

Observability is a Structural Property

$$\left\{\begin{array}{l} \dot{x} = Ax \\ y = Cx \end{array}\right. \xrightarrow[x=Mx_*]{} \left\{\begin{array}{l} x_* = M^{-1}AM\ x_* \\ y = CM\ x_* \end{array}\right.$$

$$\begin{cases} \text{Observable} \\ Q_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \\ \det\left[Q_O^T Q_O\right] \neq 0 \end{cases} \Longrightarrow \begin{cases} Q_O = \begin{bmatrix} CM \\ C\underbrace{MM^{-1}}_{I}AM \\ C\underbrace{MM^{-1}}_{I}A\underbrace{MM^{-1}}_{I}AM \\ \vdots \end{bmatrix} \\ = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} M = Q_O M \end{cases}$$

$$\det\left[Q_O^T Q_O\right] = \det\left[M^T Q_O^T Q_O M\right] = \det^2[M] \det\left[Q_O^T Q_O\right] \neq 0$$

IV.4) Test for Diagonal State-Space Model

Observability Criterion for a Single-Output System (Diagonalizable)

In that case, there exists a state-vector such that the state model writes

$$x = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \qquad \lambda_i \neq \lambda_j \text{ for } i \neq j \qquad (3)$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} x$$

The Kalman matrix is given by

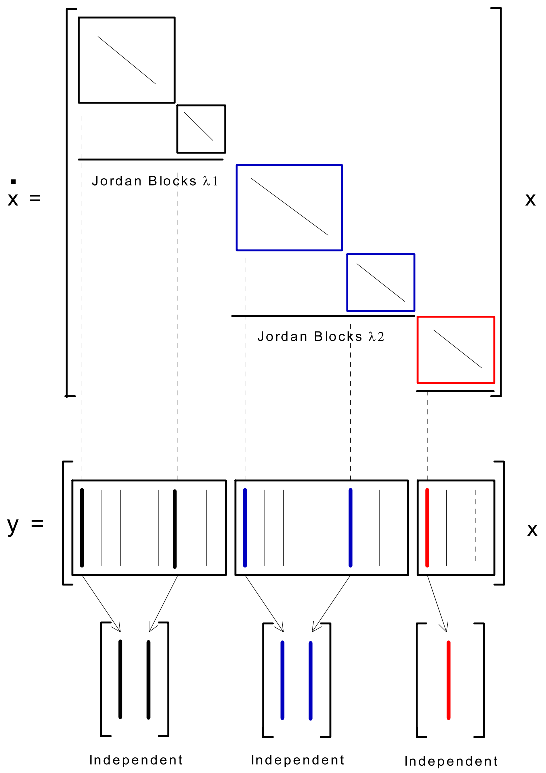
$$Q_O = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_1\lambda_1 & c_2\lambda_2 & c_3\lambda_3 & \cdots & c_n\lambda_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_1\lambda_1^{n-1} & c_2\lambda_2^{n-1} & c_3\lambda_3^{n-1} & \cdots & c_n\lambda_n^{n-1} \end{bmatrix}$$

$$Q_O = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{\text{Vandermonde Matrix invertible if } \lambda_i \neq \lambda_j \text{ for } i \neq j} \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}$$

Theorem

System (3) is observable if and only if all the $c_i \neq 0, \quad i = 1, \cdots, n.$

Generalization: Observability Criterion for a Multi-Ouputs System



Theorem

System in a canonical Jordan Form is observable if and only if the columns of the matrix C*, corresponding to the first columns of the Jordan blocks associated with a the same eigenvalue, are linearly independent.*

We can also conclude for a single-input, single-output system: if there are several Jordan blocks associated with a same eigenvalue, this one is neither controllable nor observable

IV.5) Example

$$A = \begin{bmatrix} 0 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & 1 & & & & \\ & & 0 & 1 & & & & \\ & & & & 1 & 1 & 0 & \\ & & & & 0 & 1 & 1 & \\ & & & & 0 & 0 & 1 & \\ & & & & & & & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \longleftarrow 0 \\ \longleftarrow 1 \\ \\ \longleftarrow 1 \\ \\ \\ \longleftarrow 1 \\ \longleftarrow 2 \end{matrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \\ 9 & 0 & 4 & \pi & 0 & 2 & 1 & 0 \end{bmatrix}$$

- Eigenvalue 0: Controllable and Observable
- Eigenvalue 1: Controllable and No Observable
- Eigenvalue 2: No Controllable and No Observable

IV.6) Others criteria

Popov-Belevitch-Hautus Tests

Theorem

System (2) is controllable if and only if

$$H_C = \begin{bmatrix} \lambda I - A & B \end{bmatrix}$$

*has **rank** n for all $\lambda \in \mathbb{C}$ (or for the n eigenvalues of A).*

Theorem

System (2) is observable if and only if

$$H_O = \begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

*has **rank** n for all $\lambda \in \mathbb{C}$ (or for the n eigenvalues of A).*

ASYMPTOTICALLY STABLE SYSTEMS

Controllability

Theorem

System(2) asymptotically stable is controllable if and only if the symmetric solution P_C of the Lyapunov equation

$$AP_C + P_C A^T + BB^T = 0$$

is positive definite.

Matrix P_C is called gramian of controllability and it can be written

$$P_C = \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

Observability

Theorem

System (2) asymptotically stable is observable if and only if the symmetric solution P_O of the Lyapunov equation

$$A^T P_O + P_O A + C^T C = 0$$

is positive definite.

Matrix P_O is the gramian of observability and writes

$$P_O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

Stabilizability and Detectability

Theorem

Theorem: *System (2) is stabilizable if and only if for all symmetric matrix* Q_s *, there exit matrix* L *and positive definite symmetric matrix* P_s*, solutions of the following Lyapunov equation*

$$(A+BL)P_s + P_s(A+BL)^T + Q_s = 0, \quad (i.e.\ (A+BL)P_s + P_s(A+BL)^T < 0)$$

Theorem

Theorem: *System (2) is detectable if and only if for all symmetric matrix* Q_d *, there exit matrix* H *and positive definite symmetric matrix* P_d*, solutions of the following Lyapunov equation*

$$(A+HC)^T P_d + P_d(A+HC) + Q_d = 0, \quad (i.e.\ (A+HC)^T P_d + P_d(A+HC) < 0)$$

IV.7) Duality

Consider

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$(A,B) \text{ controllable} \iff \text{Rank}\left(\underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{Q_C}\right) = n$$

But this is equivalent to

$$\text{Rank}\,[Q_C] = \text{Rank}\,[Q_C^T] = n$$

And

$$Q_C^T = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix}$$

Then we conclude

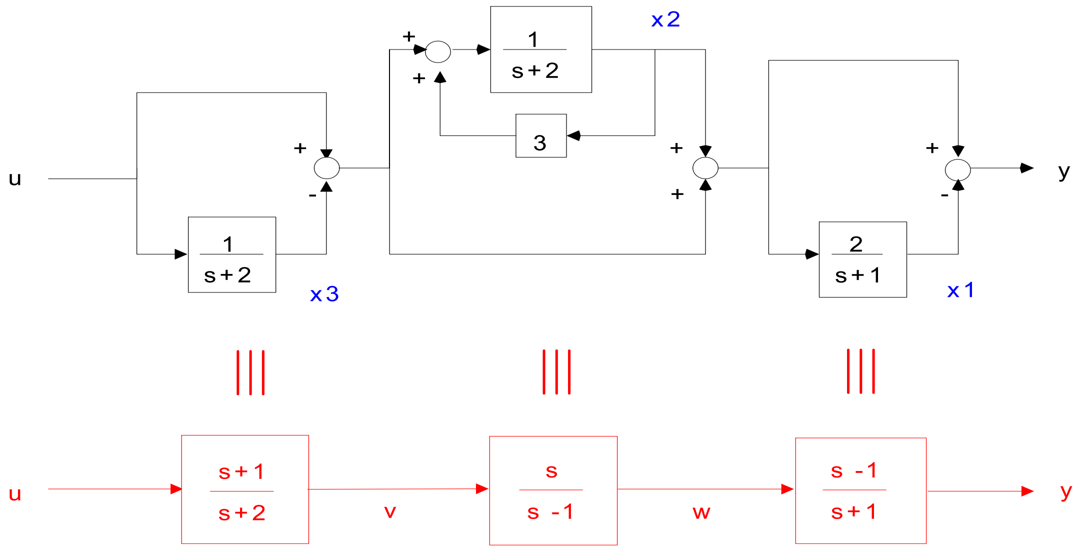
$$(A,B) \text{ Controllable} \iff (B^T, A^T) \text{ Observable}$$

$$(C,A) \text{ Observable} \iff (A^T, C^T) \text{ Controllable}$$

- We say that Controllability and Observability are dual.

IV.8) Models and Structures

Consider the following system



State-Space Model

After some elementary calculations, we obtain

$$\left\{\begin{array}{lcl} \dot{x} & = & \begin{bmatrix} -1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u \\ y & = & \begin{bmatrix} -1 & 1 & -1 \end{bmatrix} x + u \end{array}\right. \quad \text{order 3}$$

Differential equation

$$\left\{\begin{array}{l} \dot{v} + 2v = \dot{u} + u \\ \dot{w} - w = \dot{v} \\ \dot{y} + y = \dot{w} - w = \dot{v} \end{array}\right.$$

$$\ddot{y} + 3\dot{y} + 2y = \ddot{u} + \dot{u} \quad \text{order 2}$$

Transfer Function

$$G(s) = \frac{s}{s+2} \qquad \text{order } 1$$

If we consider the following state-vector

$$x = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} x_*$$

The state-space model becomes

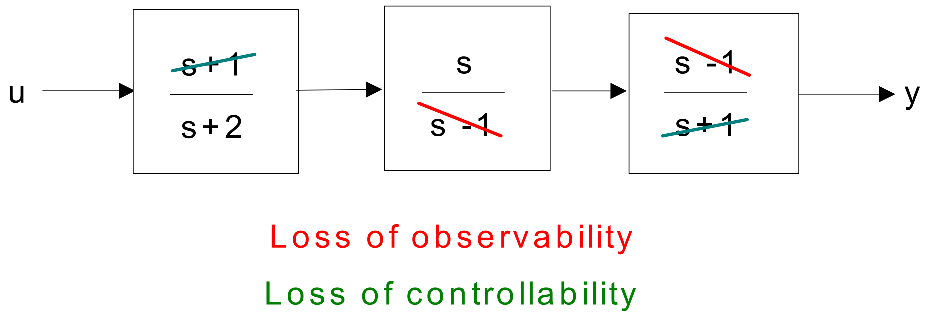
$$\begin{cases} \dot{x}_* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x_* + \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix} u \\ y = \begin{bmatrix} -1 & 0 & -6 \end{bmatrix} x_* + u \end{cases}$$

We see that mode -1 is not controllable but it is observable. 1 is controllable but not observable. -2 is controllable and observable. It can be shown that

Conclusion

- The state-space model is able to exhibit the controllable, no controllable, observable, no observable modes. A canonical representation isolating each system component can be derived. It is known as "*the canonical Kalman*" decomposition
- Differential equation exhibits the observable modes (controllable or not)
- Transfer matrix only exhibits the controllable and observable modes. For this reason, it is qualified as a "*minimal representation*" of a system

It is possible to identy the no controllable or no observable modes by inspection of poles/zeros cancellations.



MULTIVARIABLE SYSTEMS

Chapter V

Models of Multivariable Systems

Objective of Chapter V

- Recall the main models and their properties in the single-input, single-output case
- Extend them to the case of multivariable systems
- Discuss the relations between them

Outline of Chapter V

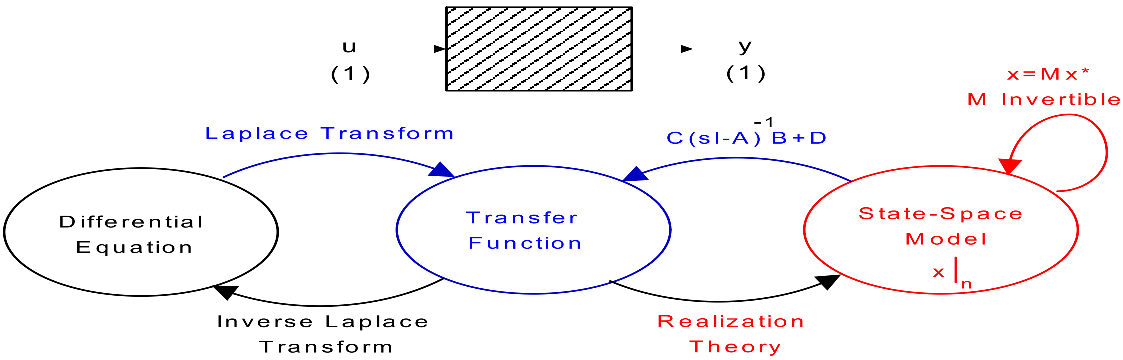
V-1. Single-Input, Single-Output Case

V-2. Multi-Input, Multi-Output Case

V-3. System of Differential Equations

V-4. Transfer Matrix

V.1) Single-Input, Single-Output Case



$$L(r)y = M(r)u$$
$$r \triangleq \frac{d}{dt}$$

$$G(s) = \frac{N(s)}{D(s)}$$
N and D relatively prime

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Characteristic Roots:
Roots of $L(r) = 0$

Poles:
Roots of $D(s) = 0$.

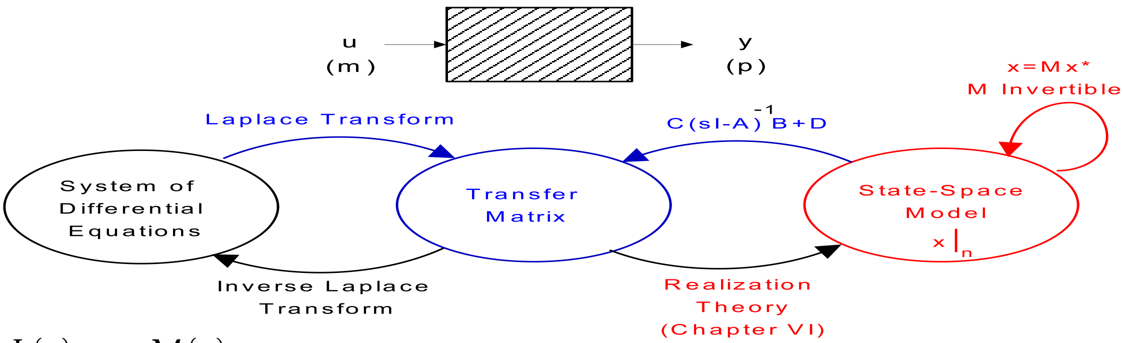
Modes :
Roots of $\det[(sI - A)] = 0$

Order $= \deg[L(r)]$

Order $= \deg[D(s)]$

Order $= \dim[x]$

V.2) Multi-Input, Multi-Output Case



$$\underset{(p\times p)}{L(r)}\,y = \underset{(p\times m)}{M(r)}\,u$$

$L(r), M(r)$ polynomial matrices

Characteristic Roots:
Roots of $\det[L(r)] = 0$

Order $= \deg\,[\det[L(r)]]$

$$\underset{(p\times m)}{G(s)} = \underset{(p\times m)}{L_R(s)}\,\underset{(m\times m)}{D_R(s)^{-1}} = \underset{(p\times p)}{D_G(s)^{-1}}\,\underset{(p\times m)}{L_G(s)}$$

Poles, Order ?
(See Chapter VI)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Modes :
Roots of $\det\,[(sI - A)] = 0$

Order $= \dim\,[x]$

V.3) System of Differential Equations

EXAMPLE 1

Consider the system described by

$$\begin{cases} \ddot{y}_1 - \dot{y}_1 + \dot{y}_2 - y_2 = u_1 + u_2 \\ \dddot{y}_1 - \ddot{y}_1 + \ddot{y}_2 + y_2 = \dot{u}_1 + \dot{u}_2 + u_2 \end{cases}$$

Letting $r \triangleq \dfrac{d}{dt}$, we can write

$$\underbrace{\begin{bmatrix} r^2 - r & r - 1 \\ r^3 - r^2 & r^2 + 1 \end{bmatrix}}_{L(r)} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & 1 \\ r & r + 1 \end{bmatrix}}_{M(r)} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u}$$

$$\det[L(r)] = \det\begin{bmatrix} r^2 - r & r - 1 \\ r^3 - r^2 & r^2 + 1 \end{bmatrix}$$
$$= (r^2 - r)(r^2 + 1) - (r - 1)(r^3 - r^2)$$
$$= r(r - 1)(r^2 + 1 - r^2 + r)$$
$$= r(r - 1)(r + 1)$$

The order is 3 and the characteristic roots are: $0, \ 1, \ -1$.

Definition

A square polynomial matrix $V(r)$ is said "unimodular" if and only its determinant is independent of r.

Definition

Consider the following systems of differential equations

$$L(r)y = M(r)u$$
$$L_1(r)y = M_1(r)u$$

They are said "*equivalent*" if and only if there exists an unimodular polynomial matrix $V(r)$ of order $p \times p$ such that

$$L_1(r) = V(r)L(r)$$
$$M_1(r) = V(r)M(r)$$

The multiplication by an unimodular matrix is equivalent to the multiplication by a scalar in the monovariable case.

EXAMPLE 2

Consider the system described by

$$\begin{cases} \ddot{y}_1 - \dot{y}_1 + 2\dot{y}_2 = u_1 + 2u_2 \\ \dot{y}_2 + y_2 = u_2 \end{cases}$$

It can be written

$$\begin{bmatrix} r^2 - r & 2r \\ 0 & r^2 + 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$\det[L(r)] = r(r-1)(r+1), \quad \text{Characteristic roots: } 0,\ 1,\ -1$$

This system is equivalent to the one of example 1. The matrix $V(r)$ is given by

$$V(r) = \begin{bmatrix} 1 & -1 \\ r & 1 - r \end{bmatrix}$$

V.4) Transfer Matrix

Take the system of Example 1

$$\begin{cases} \ddot{y}_1 - \dot{y}_1 + \dot{y}_2 - y_2 = u_1 + u_2 \\ \dddot{y}_1 - \ddot{y}_1 + \ddot{y}_2 + y_2 = \dot{u}_1 + \dot{u}_2 + u_2 \end{cases}$$

$$\begin{bmatrix} s^2 - s & s - 1 \\ s^3 - s^2 & s^2 + 1 \end{bmatrix} \underbrace{\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}}_{Y(s)} = \begin{bmatrix} 1 & 1 \\ s & s + 1 \end{bmatrix} \underbrace{\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}}_{U(s)}$$

$$
\begin{aligned}
G(s) &= \frac{1}{s(s-1)(s+1)} \begin{bmatrix} s^2+1 & 1-s \\ s^2-s^3 & s^2+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ s & s+1 \end{bmatrix} \\
&= \frac{1}{s(s-1)(s+1)} \begin{bmatrix} s+1 & 2 \\ 0 & s(s-1) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s(s-1)} & \frac{2}{s(s-1)(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}
\end{aligned}
$$

How to deduce the order, poles and zeros from the transfer matrix? Next Chapter

MULTIVARIABLE SYSTEMS

Chapter VI

State-Space Realization of Tranfer Matrices

Objective of Chapter VI

- Recall the realization theory for single-input, single-output systems
- Extend the theory to the case of multi-input, multi-output systems

Outline of Chapter VI

VI-1. Single-input, Single -Output Case

- VI-1-1. Diagonal form
- VI-1-2. Jordan Form
- VI-1-3. Companion forms

VI-2. Multi-input, Multi-Output Case

- VI-2-1. Gilbert's Method
- VI-2-2. Method of invariants : Smith-McMillan Canonical Form
- VI-2-3. Method by a Reduction of a Realization

VI-1. Single-input, Single -Output Case

Consider the system :

$$G(s) = \frac{N(s)}{D(s)} \text{ avec } \deg(N(s)) < \deg(D(s))$$

The objective is to obtain a state-space model :

$$\begin{cases} \frac{dx}{dt} & = & Ax + Bu \\ y & = & Cx \end{cases}$$

For the proper systems ($D \neq 0$), we have :

$$G(s) = \frac{N(s)}{D(s)} = \underbrace{\frac{R(s)}{D(s)}}_{(A,B,C)} + \underbrace{Q}_{D}, \quad Q : \text{quotient}, \quad R(s) : \text{rest}$$

Example:

$$G(s) = \frac{2s^2 + 7s + 7}{s^2 + 3s + 2} = \frac{2(s^2 + 3s + 2)}{s^2 + 3s + 2} + \frac{s + 3}{s^2 + 3s + 2}$$

$$\begin{cases} \dfrac{dx}{dt} & = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y & = \begin{bmatrix} 3 & 1 \end{bmatrix} x + 2u \end{cases}$$

VI-1-1. Diagonal Form

Simple poles → Partial fractions expansion.

Example:

$$G(s) = \frac{s + 3}{s^2 + 3s + 2} = \frac{2}{s + 1} - \frac{1}{s + 2}$$

$$\begin{cases} \dfrac{dx}{dt} & = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y & = \begin{bmatrix} 2 & -1 \end{bmatrix} x \end{cases}$$

VI-1-2. Jordan Form

Multiple poles $\rightarrow$ Jordan Form

Consider the system:

$$G(s) = \frac{N_1(s)N_2(s)}{(s-\lambda)^n}$$

In that case, we have:

$$A = \begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \qquad B = \begin{bmatrix} \vdots \\ \frac{\ddot{N}_2(\lambda)}{2!} \\ \frac{\dot{N}_2(\lambda)}{1!} \\ N_2(\lambda) \end{bmatrix}$$

$$C = \begin{bmatrix} N_1(\lambda) & \frac{\dot{N}_1(\lambda)}{1!} & \frac{\ddot{N}_1(\lambda)}{2!} & \cdots \end{bmatrix}$$

Example:

$$G(s) = \frac{(s+2)(s+3)}{(s+1)^5} = \frac{\overbrace{(s+2)}^{N_1(s)}\overbrace{(s+3)}^{N_2(s)}}{s^5+5s^4+10s^3+10s^2+5s+1}$$

$\lambda = -1$, we have:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Considering $N_1 = 1$ et $N_2(s) = s^2 + 5s + 6$, we have:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

VI-1-3. Companion Forms

Consider the system:

$$G(s) = \frac{N_1(s)N_2(s)}{D(s)}$$

with :

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Horizontal Companion Form

It is possible to show that:

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad B = \mathcal{A}^{-1} \begin{bmatrix} \vdots \\ \frac{\ddot{N}_2(0)}{2!} \\ \frac{\dot{N}_2(0)}{1!} \\ N_2(0) \end{bmatrix}$$

$$C = \begin{bmatrix} N_1(0) & \frac{\dot{N}_1(0)}{1!} & \frac{\ddot{N}_1(0)}{2!} & \cdots \end{bmatrix}$$

with :

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & \ddots & \cdots & 1 & 0 \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{bmatrix}$$

Example:

$$G(s) = \frac{s^2 + 3s + 2}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}}_{\mathcal{A}^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{-2} & 1 & 0 & 0 \\ \boxed{1} & \boxed{-2} & 1 & 0 \\ \boxed{0} & \boxed{1} & \boxed{-2} & 1 \end{bmatrix} = I$$

$$(1) \rightarrow \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \boxed{\star} \\ \star \\ \star \end{bmatrix} = 0 \qquad (2) \rightarrow \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ \boxed{\star} \\ \star \end{bmatrix} = 0$$

$$(3) \rightarrow \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ \boxed{\star} \end{bmatrix} = 0$$

Then we have:

$$N_1(s) = s^2 + 3s + 2 \text{ et } N_2(s) = 1 \Rightarrow C = \begin{bmatrix} 2 & 3 & 1 & 0 \end{bmatrix} \text{ et } B^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_1(s) = s + 2 \text{ et } N_2(s) = s + 1 \Rightarrow C = \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix} \text{ et } B^T = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$$

$$N_1(s) = s + 1 \text{ et } N_2(s) = s + 2 \Rightarrow C = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \text{ et } B^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$N_1(s) = 1 \text{ et } N_2(s) = s^2 + 3s + 2 \Rightarrow C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \text{ et } B^T = \begin{bmatrix} 0 & 1 & 1 & -3 \end{bmatrix}$$

Vertical Companion Form

$$A = \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \vdots \\ \frac{\ddot{N}_2(0)}{2!} \\ \frac{\dot{N}_2(0)}{1!} \\ N_2(0) \end{bmatrix}$$

$$C = \begin{bmatrix} N_1(0) & \frac{\dot{N}_1(0)}{1!} & \frac{\ddot{N}_1(0)}{2!} & \cdots \end{bmatrix} \mathcal{A}^{-1}$$

Example:

$$G(s) = \frac{s^2 + 3s + 2}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -5 & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}}_{\mathcal{A}^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We also have:

$$N_1(s) = s^2 + 3s + 2 \text{ et } N_2(s) = 1 \Rightarrow B^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \text{ et } C = \begin{bmatrix} -3 & 1 & 1 & 0 \end{bmatrix}$$

$$N_1(s) = s + 2 \text{ et } N_2(s) = s + 1 \Rightarrow B^T = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \text{ et } C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$N_1(s) = s + 1 \text{ et } N_2(s) = s + 2 \Rightarrow B^T = \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \text{ et } C = \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}$$

$$N_1(s) = 1 \text{ et } N_2(s) = s^2 + 3s + 2 \Rightarrow B^T = \begin{bmatrix} 0 & 1 & 3 & 2 \end{bmatrix} \text{ et } C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

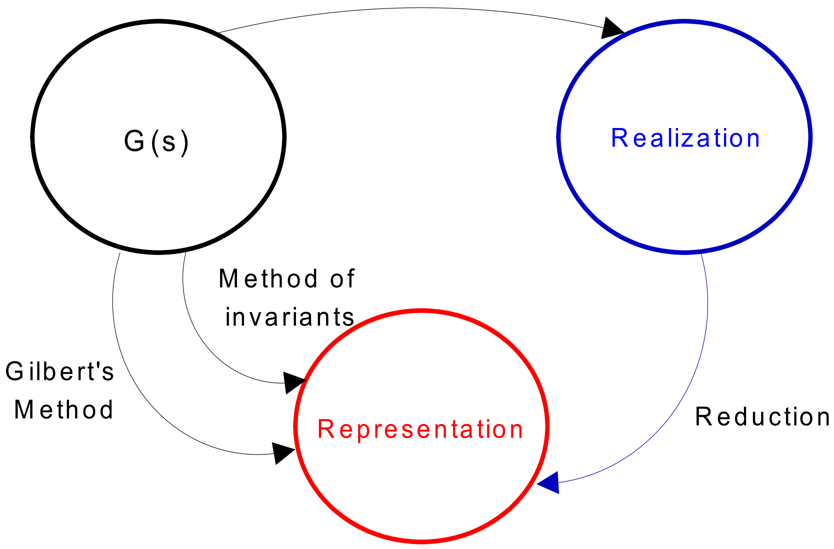
VI-2) Multi-Input, Multi-Output Case

Consider a system whose transfer matrix is $G(s)$. we refer as a:

State Realisation: A triplet (A, B, C) such that $G(s) = C(sI - A)^{-1}B$ where A *has a non minimal dimension*.

State Representation: A triplet (A, B, C) such that $G(s) = C(sI - A)^{-1}B$ where A has a *minimal dimension*.

There exist several methods to derive a state representation summarized in the following figure:



VI-2-1. Gilbert's Method

Assumption : All the roots of te denominator of G(s) are real and simple. For multivariable systems these roots are not in general the poles of the system. We can wrire :

$$G(s) = \frac{M(s)}{\psi(s)}$$

$$M(s) \text{ : Polynomial matrix } p \times m \text{ et } \psi(s) \text{ : common denominator}$$

and we have:

$$G(s) = \frac{M(s)}{\psi(s)} = \sum_{i=1}^{n} \frac{M_i}{s - \lambda_i}$$

The poles are the λ_i et their multiplicity orders are equal to the $rank(M_i)$, $i = 1, \ldots, n$. We also have :

$$\text{System order} = n = \sum_{i=1}^{n} rank(M_i)$$

Example :

$$G(s) = \frac{\begin{bmatrix} 2(2s^2+4s+1) & s(s+2) \\ 2s(3s+5) & (s+2)(3s+1) \end{bmatrix}}{s(s+1)(s+2)}$$

$$= \frac{\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{M_1}}{s} + \frac{\overbrace{\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}}^{M_2}}{s+1} + \frac{\overbrace{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}^{M_3}}{s+2}$$

The system order is:

$$\begin{aligned} n &= rang(M_1) + rang(M_2) + rang(M_3) \\ &= 2+1+1 = 4 \end{aligned}$$

We can write:

$$Y_1(s) = \frac{\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} U(s)}^{X_1}}{s} + \frac{\overbrace{\begin{bmatrix} 2 & 1 \end{bmatrix} U(s)}^{X_2}}{s+1} + \frac{\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} U(s)}^{X_3}}{s+2}$$

$$Y_2(s) = \frac{\overbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} U(s)}^{X_4}}{s} + \frac{\overbrace{\begin{bmatrix} 4 & 2 \end{bmatrix} U(s)}^{X_5}}{s+1} + \frac{\overbrace{\begin{bmatrix} 2 & 0 \end{bmatrix} U(s)}^{X_6}}{s+2}$$

$$X_i(s) = \frac{\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}}{s+c}$$

$$sX_i(s) = -cX_i(s) + \begin{bmatrix} a & b \end{bmatrix} \underbrace{\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}}_{U(s)}$$

$$\downarrow$$

$$\dot{x}_i = -c\, x_i + \begin{bmatrix} a & b \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \\ 4 & 2 \\ 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

We can also write :

$$Y_1(s) = \frac{\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} U(s)}^{x_1}}{s} + \frac{\overbrace{\begin{bmatrix} 2 & 1 \end{bmatrix} U(s)}^{x_2}}{s+1} + \frac{\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} U(s)}^{x_3}}{s+2}$$

$$Y_2(s) = \frac{\overbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} U(s)}^{x_4}}{s} + \frac{\overbrace{\begin{bmatrix} 4 & 2 \end{bmatrix} U(s)}^{2x_2}}{s+1} + \frac{\overbrace{\begin{bmatrix} 2 & 0 \end{bmatrix} U(s)}^{2x_3}}{s+2}$$

And we have:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

Another possibility is:

$$Y_1(s) = \frac{\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} U(s)}^{x_1}}{s} + \frac{\overbrace{\begin{bmatrix} 2 & 1 \end{bmatrix} U(s)}^{\frac{1}{2}x_3}}{s+1} + \frac{\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} U(s)}^{\frac{1}{2}x_4}}{s+2}$$

$$Y_2(s) = \frac{\overbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} U(s)}^{x_2}}{s} + \frac{\overbrace{\begin{bmatrix} 4 & 2 \end{bmatrix} U(s)}^{x_3}}{s+1} + \frac{\overbrace{\begin{bmatrix} 2 & 0 \end{bmatrix} U(s)}^{x_4}}{s+2}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 2 \\ 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

VI-2-2. Method of invariants : Smith-McMillan Canonical Form

We consider the system described by:

$$G(s) = \frac{M(s)}{\psi(s)}$$

$M(s)$:Polynomial matrix $p \times m$ et $\psi(s)$: common denominator

$S(s)$ et $M(s)$ are said equivalent if there exist two unimodular matrices (of constant determinant independent of s) such that:

$$\underbrace{M(s)}_{p \times m} = \underbrace{V(s)}_{p \times p} \underbrace{S(s)}_{p \times m} \underbrace{W(s)}_{m \times m}$$

A particular important case where $S(s)$ is pseudo-diagonal :

$$S(s) = \begin{bmatrix} \gamma_1(s) & & & & & \\ & \ddots & & & & \\ & & \gamma_r(s) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

This is the Smith form of $M(s)$ and the rank of $M(s) = r$.

The $\gamma_i(s)$ are the invariants of $S(s)$.

Two methods exist to compute $S(s)$:

- Pseudo-diagonalisation algorithm allowing to find simultaneously $V(s),\ S(s)$ et $W(s)$.
- A direct method allowing to compute $S(s)$, but not $V(s)$ et $W(s)$.

The direct method is as follows :

1. Let $\Delta_0 = 1$ and define $\Delta_i(s)$ as the greatest common divisor (GCD) of the minors of $S(s)$ of order i.
2. We have:

$$\gamma_1(s) = \frac{\Delta_1(s)}{\Delta_0(s)}, \quad \gamma_2(s) = \frac{\Delta_2(s)}{\Delta_1(s)}, \cdots, \quad \gamma_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)}, \cdots$$

Note that $\gamma_i(s)$ divides $\gamma_{i+1}(s)$.

Example :

Consider the polynomial matrix:

$$M(s) = \begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2(s^2-4) \end{bmatrix}$$

The structure of $S(s)$ is a priori the following:

$$S(s) = \begin{bmatrix} \gamma_1(s) & 0 \\ 0 & \gamma_2(s) \\ 0 & 0 \end{bmatrix}$$

Run the algorithm:

1. $\Delta_0 = 1$
2. The minors of order 1 are the elements of $M(s)$. Their GCD is equal to $\Delta_1 = 1$. The minors of orders 2 are:

$$3s^2 - 12$$
$$3s^2 - 12$$
$$3s^3 - 12s$$

Their GCD is $\Delta_2(s) = 3s^2 - 12$. Then we have :

$$\gamma_1(s) = \frac{\Delta_1}{\Delta_0} = 1, \qquad \gamma_2(s) = \frac{\Delta_2}{\Delta_1} = 3(s^2 - 4)$$

$$\text{et } S(s) = \begin{bmatrix} 1 & 0 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}$$

Taking into account the previous developments, we have:

$$G(s) = \frac{V(s)S(s)W(s)}{\psi(s)} = V(s)\frac{S(s)}{\psi(s)}W(s)$$

The matrix $\dfrac{S(s)}{\psi(s)}$ is a diagonal matrix composed of polynomial fractions.

The diagonal involves polynomial fractions whose numerator and denominator have to be relatively prime.

If common factors appear between numerators and denominators, they have to be simplified.

Denoting:

$$V(s) = \begin{bmatrix} V_1(s) & \cdots & V_p(s) \end{bmatrix} \quad W(s) = \begin{bmatrix} W_1(s) \\ \vdots \\ W_m(s) \end{bmatrix}$$

$$\frac{S(s)}{\psi(s)} = \begin{bmatrix} \frac{\epsilon_1(s)}{\psi_1(s)} & & & & & \\ & \ddots & & & & \\ & & \frac{\epsilon_r(s)}{\psi_r(s)} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

we can write :

$$G(s) = \frac{V(s)S(s)W(s)}{\psi(s)} = V(s)\frac{S(s)}{\psi(s)}W(s)$$
$$= \sum_{i=1}^{r} V_i(s)\frac{\epsilon_i(s)}{\psi_i(s)}W_i(s)$$

Then:

The poles of $G(s)$ are the roots of $\psi_i(s)$
The zeros of $G(s)$ are the roots of $\epsilon_i(s)$

The system order is equal to $n = \sum_{i=1}^{r} \deg(\psi_i(s))$

Example

Consider the system described by :

$$G(s) = \frac{\begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2(s^2-4) \end{bmatrix}}{s(s+2)(s+1)}$$

We have:

$$\frac{\epsilon_1(s)}{\psi_1(s)} = \frac{\gamma_1(s)}{\psi(s)} = \frac{1}{s(s+2)(s+1)}$$

$$\frac{\epsilon_2(s)}{\psi_2(s)} = \frac{\gamma_2(s)}{\psi(s)} = \frac{3(s^2-4)}{s(s+2)(s+1)} = \frac{3(s-2)}{s(s+1)}$$

And:

$$G(s) = V(s) \begin{bmatrix} \frac{1}{s(s+2)(s+1)} & 0 \\ 0 & \frac{3(s-2)}{s(s+1)} \\ 0 & 0 \end{bmatrix} W(s) \quad \textit{(Smith-MacMillan Canonical Form)}$$

- The system order is: 3+2=5
- 0 is a pole of multiplicity 2
- -1 is a pole of multiplicity 2
- -2 is a simple pole. 2 is a simple zero.

State-Space Representation from the Smith Mc-Millan Form

Suppose that we know the matrices $V(s)$ et $W(s)$,. We can write:

$$Y(s) = G(s)U(s) = \sum_{i=1}^{r} V_i(s)\frac{\epsilon_i(s)}{\psi_i(s)}W_i(s)U(s)$$

The state-space representation is deduced as the sum of each term:

$$y_i = V_i(s)\frac{\epsilon_i(s)}{\psi_i(s)}W_i(s)U(s) = \frac{Z_i(s)W_i(s)}{\psi_i(s)}U(s)$$

We merge $\epsilon_i(s)$ to $V_i(s)$ or to $W_i(s)$. Adapting the results of paragraph VI.1, we deduce:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u \\ y_i = C_i x_i \end{cases}$$

The representation is obtained aggregating all the terms in the following way:

$$\begin{cases} \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{bmatrix} X + \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix} u \\ \\ Y = \begin{bmatrix} C_1 & \cdots & C_r \end{bmatrix} X \end{cases}$$

Example :

Consider the system described by:

$$\begin{bmatrix} 1 & 0 & 0 \\ s^2+s-4 & 1 & 0 \\ s^2-1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s(s+2)(s+1)} & 0 \\ 0 & \dfrac{3(s-2)}{s(s+1)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 \\ s^2+s-4 \\ s^2-1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}}{s^3+3s^2+2s} + \frac{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} 3(s-2) \begin{bmatrix} 0 & 1 \end{bmatrix}}{s^2+s}$$

Consider the first term:

$$\frac{\overbrace{\begin{bmatrix} 1 \\ s^2+s-4 \\ s^2-1 \end{bmatrix}}^{Z_1(s)} \overbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}^{W_1(s)}}{s^3+3s^2+2s} \Rightarrow \mathcal{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \quad \mathcal{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{bmatrix}$$

Then:

$$\begin{cases} \dot{x}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u \\ y_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} x_1 \end{cases}$$

Now consider the second one:

$$\frac{\overbrace{\begin{bmatrix} 0 \\ 3(s-2) \\ -3(s-2) \end{bmatrix}}^{Z_2(s)} \overbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}^{W_2(s)}}{s^2+s} \Rightarrow \mathcal{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathcal{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Then :

$$\begin{cases} \dot{x}_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} u \\ y_2 = \begin{bmatrix} 1 & 0 \\ -6 & 3 \\ 6 & -3 \end{bmatrix} x_2 \end{cases}$$

And the representation is given by:

$$\begin{cases} \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & -2 & -3 & & \\ & & & 0 & 1 \\ & & & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \\ Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ -4 & 1 & 1 & -6 & 3 \\ -1 & 0 & 1 & 6 & -3 \end{bmatrix} X \end{cases}$$

VI-2-3. Method by a Reduction of a Realization

Consider the system described by:

$$G(s) = \frac{M(s)}{\psi(s)} = \frac{\left[\begin{array}{ccc} M^1(s) & \cdots & M^m(s) \end{array}\right]}{\psi(s)}$$
$$= \frac{M^1(s)\left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]}{\psi(s)} + \frac{M^2(s)\left[\begin{array}{cccc} 0 & 1 & \cdots & 0 \end{array}\right]}{\psi(s)}$$
$$+ \cdots + \frac{M^m(s)\left[\begin{array}{cccc} 0 & 0 & \cdots & 1 \end{array}\right]}{\psi(s)}$$

For each term:

$$y_i = \frac{M^i(s)\left[\begin{array}{ccccc} 0 & \cdots & 1 & \cdots & 0 \end{array}\right]}{\psi(s)} U(s)$$

We can obtain by a method proposed in Paragraph VI-1:

$$\begin{cases} \dot{x}_i = A_i x_i + B_1 u \\ y_i = C_i x_i \end{cases}$$

And representation for the overall system can be deduced as:

$$\begin{cases} \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{bmatrix} X + \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} u \\ \\ Y = \left[\begin{array}{ccc} C_1 & \cdots & C_m \end{array}\right] X \end{cases}$$

Minimality Test

1. Check the observability
2. If the representation is observable, in fact ot is a realization
3. If not, we can use the reduction techniques proposed, for example by MATLAB (function [it minreal).

Examples : Consider the transfer matrix:

$$G(s) = \frac{\begin{bmatrix} 3s+1 & 7s^2+4s \\ s-1 & 1 \end{bmatrix}}{(s-2)^3}$$

$$= \frac{\begin{bmatrix} 3s+1 \\ s-1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{(s-2)^3} + \frac{\begin{bmatrix} 7s^2+4s \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{(s-2)^3}$$

$$\begin{cases} \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & & & \\ 0 & 2 & 1 & & & \\ 0 & 0 & 2 & & & \\ & & & 2 & 1 & 0 \\ & & & 0 & 2 & 1 \\ & & & 0 & 0 & 2 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \boxed{1} & \boxed{0} \\ 0 & 0 \\ 0 & 0 \\ \boxed{0} & \boxed{1} \end{bmatrix} u \\ Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} \boxed{7} & 3 & 0 & \boxed{36} & 32 & 7 \\ \boxed{1} & 1 & 0 & \boxed{1} & 0 & 0 \end{bmatrix} X \end{cases}$$

This representation is observable, then it is a realization.

$$G(s) = \frac{\begin{bmatrix} s^3 - s^2 + 1 & 1 & -s^3 + s^2 - 2 \\ 1.5s + 1 & s + 1 & -1.5s - 2 \\ s^3 - 9s^2 - s + 1 & -s^2 + 1 & s^3 - s - 2 \end{bmatrix}}{s^4}$$

$$A = \begin{bmatrix} 0 & 1 & & & & & & & & & & \\ & 0 & 1 & & & & & & & & & \\ & & 0 & 1 & & & & & & & & \\ & & & 0 & & & & & & & & \\ & & & & 0 & 1 & & & & & & \\ & & & & & 0 & 1 & & & & & \\ & & & & & & 0 & 1 & & & & \\ & & & & & & & 0 & & & & \\ & & & & & & & & 0 & 1 & & \\ & & & & & & & & & 0 & 1 & \\ & & & & & & & & & & 0 & 1 \\ & & & & & & & & & & & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \boxed{1} & 0 & -1 & 1 & \boxed{1} & 0 & 0 & 0 & \boxed{-2} & 0 & 1 & -1 \\ \boxed{1} & 1.5 & 0 & 0 & \boxed{1} & 1 & 0 & 0 & \boxed{-2} & -1.5 & 0 & 0 \\ \boxed{1} & -1 & -9 & 1 & \boxed{1} & 0 & -1 & 0 & \boxed{-2} & -1 & 0 & 1 \end{bmatrix}$$

This representation is not observable. It must be reduced using, for example, the MATLAB procedure *minreal*.

MULTIVARIABLE SYSTEMS

Chapter VII

Linear Quadratic Optimal Control

Problem statement Linear optimal Control: State feedback Asymptotic Properties Phase and gain margins Some Extensions Examples

Objective of Chapter VII

- Introduce the linear quadratic optimal control problem (LQR State Feedback)
- Discuss important properties (asymptotic properties, phase and gain margins)
- Discuss some numerical issues

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2. Linear optimal Control: State feedback
3. Asymptotic Properties
4. Phase and gain margins
5. Some Extensions
6. Examples

VII.1 - Problem statement

- Consider the system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{cases}$$

- The state is supposed measurable
- Associated with the system, define the following criterion

$$J(x,u) = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt$$

avec

$$R > 0 \text{ et } Q = Q^T \geqslant 0$$

- (Recall that $P = P^T > 0(\geqslant) \Rightarrow x^TPx > 0(\geqslant) \ \ \forall x \neq 0$)

VII.1 - Problem statement

- The criterion is of energy type. In fact if $Q = I$ and $R = I$, it represents the sum of state and input energies
- If $Q = C^T C$ and $R = I$, it represents the sum of output and input energies

We consider the problem

Problem

$$\begin{cases} \min\limits_{u(t)} J(x,u) \\ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \end{cases} \tag{1}$$

To solve the problem, several approaches exist.

- Calculus of variations
- Dynamic Programming
- Pontryagin Maximum Principle

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VII.2 - Linear optimal Control: State feedback

The solution of Problem 1 is obtained by the following procedure

Procedure

1. Choose $Q \geqslant 0$ et $R > 0$ with $(A^T, Q^{1/2})$ controllable.
2. Determine the positive definite matrix P solution of the following algebraic Riccati equation
$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$
3. The optimal control law is given by :
$$u(t) = -Kx(t) = -R^{-1}B^TPx(t)$$
4. The optimal value of the performance index is:
$$J^* = x_0^TPx_0$$

VII.2 - Linear optimal Control: State feedback

- The numerical determination of matrix P is simple. Under classical assumptions, it can be obtained diagonalizing the Hamiltonian matrix associated with the previous Riccati equation expressed as
$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$
(This matrix has n eigenvalue λ_i such that $\Re e[\lambda_i] < 0$ and n eigenvalues with $\Re e[\lambda_i] > 0$. Taking the n eigenvectors associated with the n stable eigenvalues denoted
$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad X_1 \in \mathbb{R}^{n\times n}, \quad X_2 \in \mathbb{R}^{n\times n}$$
we have $P = X_2X_1^{-1}$.)
- The optimal control is a state feedback. Matrices Q and R can be used to manage the tradeoff between control effort and the associated states dynamics.

VII.2 - Linear optimal Control: Stability

An important property is that for all matrices R and Q satisfying the considered assumptions, the closed-loop system will be asymptotically stable. Indeed

$$A^T P + PA - PBR^{-1}B^T P + Q =$$
$$(A - BR^{-1}B^T P)^T P + P(A - BR^{-1}B^T P) + PBR^{-1}\, R\, R^{-1}B^T P + Q =$$
$$(A - BK)^T P + P(A - BK) + Q + K^T RK = 0$$

And then we conclude the the closed loop system is asymptotically stable because

$$\boxed{(A - BK)^T P + P(A - BK) = -(Q + K^T RK) < 0} \qquad (2)$$

(In addition, invoking Lyapunov theory for linear systems, we have

$$J = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt = \int_0^\infty x^T(t)\left[Q + K^T RK\right] x(t)dt$$
$$= x_0^T \left[\int_0^\infty e^{(A-BK)^T t}\left[Q + K^T RK\right] e^{(A-BK)t} dt\right] x_0 = x_0^T P x_0$$

where P is the solution of (2).)

VII.2 - Linear optimal Control: Optimality

- If we admit that the optimal control law is a state feedback, then K obtained through the positive definite solution of the Riccati equation is optimal.
- Consider $K + \Delta K$ and the associated variation of P, $P + \Delta P$. We have

$$[A - B(K + \Delta K)]^T (P + \Delta P) + (P + \Delta P)[A - B(K + \Delta K)] +$$
$$Q + (K + \Delta K)^T R(K + \Delta K) = 0$$

Subtracting (2) and remarking that :

$$-\Delta K^T B^T P = -\Delta K^T RK, \quad (-\Delta K^T R \overbrace{R^{-1}B^T P}^{K})$$

And then:

$$[A - B(K + \Delta K)]^T \Delta P + \Delta P[A - B(K + \Delta K)] + \Delta K^T R \Delta K = 0$$

- If ΔK is such that the matrix $A - B(K + \Delta K)$ is unstable, the performance index is infinite
- If ΔK is such that the matrix $A - B(K + \Delta K)$ is stable, invoking Lyapunov stability results, we have $\Delta P \geqslant 0$.

In all the case, the value of performance index is superior

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VII.3 - Asymptotic Properties

- Is it possible to relate optimal control (LQR) to pole placement?
- The answer is not simple. There exists a lot of works dealing with the relation between the selection of weighting matrices and pole placement

If $R = rI$ and given Q,

> When $r \to \infty$
>
> - The stable eigenvalues of matrix A are also eigenvalues of $A - BK$
> - If λ is an unstable eigenvalue of matrix A, then $-\lambda$ is an eigenvalue of $A - BK$. This is called the *"mirror effect"*.

We can conclude that stabilize an unstable system with an input of minimum energy can be done putting closed-loop poles at the mirror images of the unstable ones

$r \to 0$

- Some closed-loop eigenvalues tend to the stable zeros of $Q^{1/2}(sI - A)^{-1}B$. some of them tend to the mirror images of unstable zeros of $Q^{1/2}(sI - A)^{-1}B$.
- The remaining eigenvalues assume a Butterworth pattern, whose radius increases to infinity. The angular separation of n closed-loop poles on the arc is constant, and equal to $\frac{180^\circ}{n}$. An angle $\frac{90^\circ}{n}$ separates the most lightly-damped poles from the imaginary axis (see figure III.1).

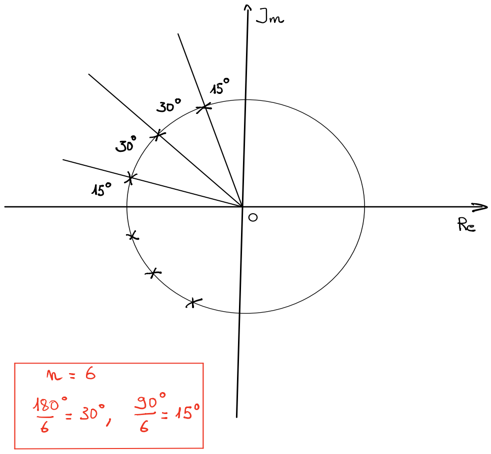

Fig.VII.1 - Butterworth Pole Configuration

- When $Q = C^TC$, the zeros of $Q^{1/2}(sI - A)^{-1}B$ are the zeros of the open-loop system $C(sI - A)^{-1}B$.
- When r is small, some elements of the control gain $K = R^{-1}B^TP$ can be very large and the control can saturate.

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VII.4 - Phase and gain margins: From Lyapunov Theory

- Suppose that the input matrix B is perturbed and is given by BG where G is a diagonal matrix of appropriate dimensions
- From the positive definite solution P of the Riccati equation, take the Lyapunov function $V(x) = x^T P x$.
- We have

$$\begin{aligned}\dot{V}(x) =& 2x^T P \dot{x} \\ =& 2x^T P(A - BGR^{-1}B^T P)x \\ =& x^T(-Q + PBR^{-1}B^T P)x - 2x^T GBR^{-1}B^T P)x, \\ & (\text{From Riccati equation, } 2x^T PAx = x^T(-Q + PBR^{-1}B^T P)x) \\ =& -x^T(Q - PB(I - 2G)R^{-1}B^T P)x\end{aligned}$$

- $\dot{V}(x) < 0$ if $Q - PB(I - 2G)R^{-1}B^T P > 0$.

> - If $G = diag(g_{11}, \ldots, g_{nn})$ then $g_{ii} \geqslant 1/2$ (infinite gain margin)
> - If $G = diag(e^{j\Delta\phi_1}, \ldots, e^{j\Delta\phi_n})$ then $|\Delta\phi_1| \leqslant 60°$

VII.4 - Phase and gain margins: Kalman Inequality

From $A^T P + PA - PBR^{-1}B^T P + Q = 0$ and all complex number s, we have

$$(\bar{s}I - A^T)P + P(sI - A) + PBR^{-1}B^T P = Q + 2\,\text{Re}[s].P$$

Multiplying on the left by $B^T(\bar{s}I - A^T)^{-1}$ and on the right by $(sI - A)^{-1}B$, we obtain

$$RG_K(s) + G_K^T(\bar{s})R + G_K^T(\bar{s})RG_K(s) = H(s)$$

with

$$\begin{aligned}H(s) &= B^T(\bar{s}I - A^T)^{-1}\,[Q + 2\,\text{Re}[s].P]\,(sI - A)^{-1}B \\ G_K(s) &= R^{-1}B^T P(sI - A)^{-1}B = K(sI - A)^{-1}B\end{aligned}$$

$G_K(s)$ is nothing else than the open-loop transfer matrix. Then

$$\boxed{\left(1 + G_K^T(\bar{s})\right) R \left(1 + G_K(s)\right) = R + H(s)}$$

If $s = j\omega$, the following inequality can be deduced (Kalman Inequality)

$$\boxed{\left(1 + G_K^T(-j\omega)\right) R \left(1 + G_K(j\omega)\right) = R + H(j\omega) \geqslant R}$$

VII.4 - Phase and gain margins: Kalman Inequality

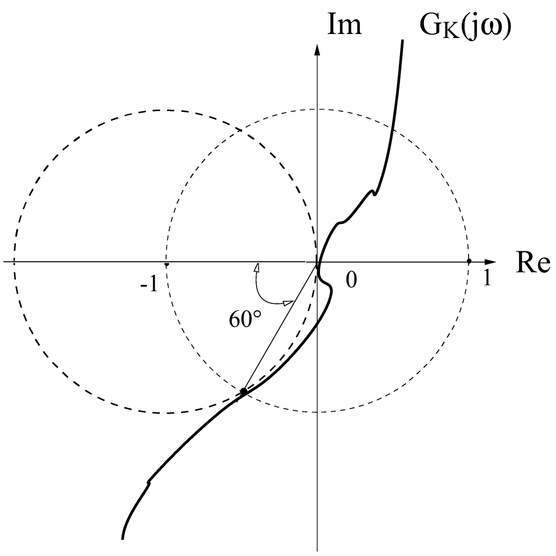

Fig.VII.2 - Gain and Phase Margins

- For a Single Input system, we obtain

$$|1 + G_K(j\omega)| \geqslant 1$$

For all ω, $G_K(j\omega)$ remains outside the circle of radius 1 centered at $(-1, 0)$

- If the open-loop transfer function $G_K(j\omega)$ is multiplied by a constant g such that $0.5 \leqslant g < \infty$, the closed-loop system is asymptotically stable
- If the open-loop transfer function $G_K(j\omega)$ is multiplied by a constant $g = e^{j\Delta\Phi}$ such that $-60° \leqslant \Delta\Phi \leqslant 60°$, the closed-loop system is asymptotically stable. The phase margin is greater that $60°$.

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VII.5 - Some Extensions: Extended index

- we can extend the performance index and consider

$$J(x,u) = \int_0^\infty \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$
$$= \int_0^\infty (x^T Q x + u^T R u + 2x^T N u)\, dt$$

with

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \geqslant 0, \quad R > 0$$

- The Riccati equation and the control become

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0, \;\; u(t) = -R^{-1}(B^T P + N^T)x(t)$$

- The associated MATLAB procedure is

$$[K, P, Closed\_loop\_eigenvalues] = lqr(A, B, Q, R, N)$$

VII.5 - Some Extensions: Finite Horizon Problem

- The finite horizon problem can stated as

Problem

$$\begin{cases} \min\limits_{u(t)} J(x,u) = \int_0^{t_f} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q(t) & N(t) \\ N^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + \frac{1}{2} x^T(t_f) Q_f x(t_f) \\ \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(0) = x_0 \textit{ and } t_f \textit{ given} \\ x(t_f) \textit{ is free and } Q_f \geqslant 0 \\ \forall t \in [0, t_f] \begin{bmatrix} Q(t) & N(t) \\ N^T(t) & R(t) \end{bmatrix} \geqslant 0, \quad R(t) > 0 \end{cases} \tag{3}$$

- The solution is given by

$$u(t) = -R(t)^{-1} \left[B^T(t)P(t) + N^T(t)\right] x(t)$$
$$\dot{P}(t) = A^T(t)P(t) + P(t)A(t) - [P(t)B(t) + N(t)]R^{-1}(t)[B^T(t)P(t) + N^T(t)] + Q(t)$$
$$P(t_f) = Q_f$$

VII.6 - Some Extensions: Tracking

- we can extend the optimal regulation to an optimal tracking problem. For simplicity, we consider the single input single output case

Problem

Consider the system

$$\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \right.$$

Let $y_c(t)$ the reference signal. The problem is to design a state-feedback stabilizing asymptotically the system and ensuring that

$$\lim_{t\to\infty} y(t) = y_c(t)$$

Problem statement | Linear optimal Control: State feedback | Asymptotic Properties | Phase and gain margins | Some Extensions | Examples

VII.6 - Some Extensions: Tracking

Suppose that there exist x_c et u_c such that

$$\left\{ \begin{array}{l} \dot{x}_c = Ax_c + Bu_c \\ y_c = Cx_c \end{array} \right. \tag{4}$$

If x_c et u_c do not exist, then the tacking problem has no solution. A necessary condition is given by

$$\boxed{\text{rang}\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right) = n + 1}$$

or equivalently

$$\boxed{\det \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \neq 0}$$

VII.65- Some Extensions: Tracking

Introduce the error signals

$$\Delta x = x - x_c$$
$$\Delta u = u - u_c$$

Then

$$\begin{cases} \dot{\Delta x} = A\Delta x + B\Delta u \\ \Delta y = C\Delta x = y - y_c \end{cases}$$

We can associate the following performance index involving Δy and Δu :

$$J_y = \int_0^\infty \left[q(\Delta y)^2 + r(\Delta u)^2\right] dt = \int_0^\infty \left[q\Delta x^T C^T C\Delta x + r(\Delta u)^2\right] dt$$

The problem is formulated a an LQR problem with weighting matrices $Q = qC^TC$ and r.
The solution is given by

$$\Delta u = -r^{-1}B^TP\Delta x = -r^{-1}B^TPx + r^{-1}B^TPx_c$$

where P is the solution of the algebraic Riccati equation

$$A^TP + PA - r^{-1}PBB^TP + C^TC = 0$$

VII.5 - Some Extensions: Tracking

We can remark that to implement the control law, we need x_c. Suppose that y_c is adifferentiable signal and that $y_c^{(l)} = 0,\ \forall l > r$. Then we can write

$$\underbrace{\begin{bmatrix} A & -I & 0 & \cdots & 0 & B & 0 & \cdots & 0 \\ 0 & A & -I & \cdots & 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & A & 0 & 0 & \cdots & B \\ C & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C & 0 & 0 & \cdots & 0 \end{bmatrix}}_{W} \begin{bmatrix} x_c \\ x'_c \\ x''_c \\ \vdots \\ x_c^{(r)} \\ u_c \\ u'_c \\ \vdots \\ u_c^{(r)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ y_c \\ y'_c \\ \vdots\ y_c^{(r)} \end{bmatrix}$$

We can show by permuting lines and columns of W that it is invertible if and only if

$$\det \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \neq 0$$

VII.5 - Some Extensions: Tracking

Procedure

1. Select $q,\ r,\ y_c$
2. Determine the solution $P > 0$ of the algebraic RIccati equation
$$A^TP + PA - r^{-1}PBB^TP + qC^TC = 0$$
3. Determine u_c et x_c from the r non null derivatives of y_c
$$\begin{bmatrix} x_c \\ \vdots \\ x_c^{(r)} \\ u_c \\ \vdots \\ u_c^{(r)} \end{bmatrix} = W^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y_c \\ \vdots \\ y_c^{(r)} \end{bmatrix}$$
4. The control law is given by
$$u(t) = u_c(t) - r^{-1}B^TPx(t) - r^{-1}B^TPx_c(t)$$

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VII.6 - Examples: Example 1

Consider the system

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

with the quadratic criterion

$$J = \int_0^{\infty} \left[y^2(t) + ru^2(t)\right] dt, \; r > 0$$

The positive define solution $P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_3 \end{bmatrix}$ satisfies the Riccati equation and then

$$\begin{aligned} 1 - r^{-1}p_2^2 &= 0 \\ p_1 - r^{-1}p_2p_3 &= 0 \\ 2p_2 - r^{-1}p_3^2 &= 0 \end{aligned}$$

The solution is given by

$$P = \begin{bmatrix} \sqrt{2\sqrt{r}} & \sqrt{r} \\ \sqrt{r} & \sqrt{2r\sqrt{r}} \end{bmatrix}$$

By the Sylvester test, we can verify that P is positive. Indeed

$$\sqrt{2\sqrt{r}} > 0 \text{ et } 2r - r = r > 0$$

The control is

$$u(t) = -Kx(t) = -\begin{bmatrix} \frac{1}{\sqrt{r}} & \sqrt{\frac{2}{\sqrt{r}}} \end{bmatrix} x(t)$$

The characteristic polynomial of $A - BK$ and poles are :

$$s^2 + \sqrt{\frac{2}{\sqrt{r}}}s + \frac{1}{\sqrt{r}} = 0, \;\; s_{1,2} = \frac{1}{\sqrt{2\sqrt{r}}} \pm j\frac{1}{\sqrt{2\sqrt{r}}}$$

When $r \to 0$, The poles modules tend to infinity because $Q^{1/2}(sI - A)C = s^{-2}$ (not finite zeros).

VII.6 - Examples: Example 2

Consider the system

$$\begin{cases} \dot{x} = \begin{bmatrix} -4 & 5 & -8 \\ 5 & -4 & 10 \\ 4 & -4 & 9 \end{bmatrix} x + \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \end{bmatrix} u \\ \\ y = \begin{bmatrix} -4 & 6 & -8 \end{bmatrix} x \end{cases}$$

The transfer function is

$$G(s) = \frac{(s-2+j)(s-2-j)}{(s-1)(s-3)(s+3)}$$

VII.6 - Examples: Example 2

Take $Q = C^TC$ and $R = r$ for $r = 0.1,\ 0.00001,\ 1,\ 10000.$ We use the MATLAB procedure

$$[K, P, Closed\_loop\_eigenvalues] = lqr(A, B, Q, r, 0)$$

r	**Control Gain**	**Closed-Loop Eigenvalues**
.00001	$K = \begin{bmatrix} 1290. & 1893. & 28. \end{bmatrix}$	$-2.00 + j1.00,\quad -2.00 - j0.00,\quad -316.2$
0.1	$K = \begin{bmatrix} 30.64 & 18.78 & 30.96 \end{bmatrix}$	$-1.87 + j0.56,\quad -1.87 - j0.56,\quad -4.76$
1	$K = \begin{bmatrix} 24.91 & 9.49 & 31.82 \end{bmatrix}$	$-1.15,\quad -2.59,\quad -3.46$
10000	$K = \begin{bmatrix} 24.00 & 8.00 & 32.00 \end{bmatrix}$	$-1.00,\quad -3.00,\quad -2.99$

We can verify the asymptotic behavior described above

VII.6 - Examples: Example 2

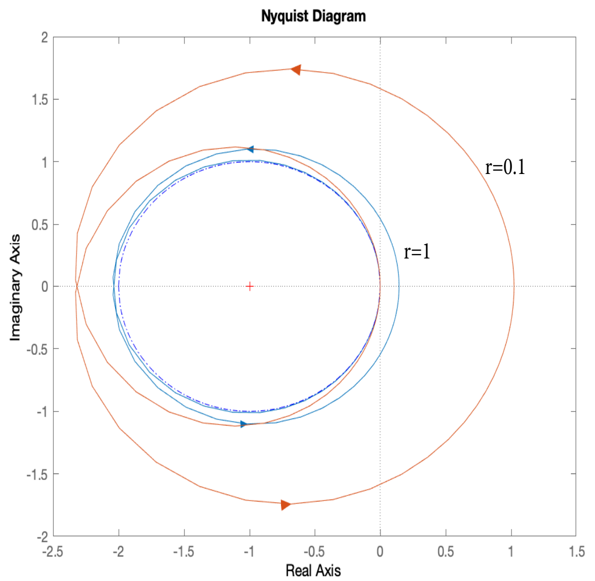

Fig.III.3 - Nyquist plot of $G_k(s) = K(sI - A)^{-1}B$

- The nyquist plot of the open-loop transfer function

$$G_k(s) = K(sI - A)^{-1}B$$

remains outside the crcle of radius 1 centered at $(-1, 0)$

- Phase margin is greater than 60°and gain margin is infinite