RANDOM SIGNALS

Chapter IV

Spectral Analysis of Random Signals

Objective of Chapter IV

- Find a way to define appropriately a spectral density for a stochastic process
- Introduce some important examples
- Discuss some extensions to special family of random signals
- Introduce some basic facts concerning filtering

1. Classification of signals from their energetic properties

- **Finite energy signal or Energy signal (L_2).** The energy of a signal $s(t)$ (s_k) is finite if

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \quad \left(E_{x_k} = \sum_{k=-\infty}^{\infty} |x_k|^2 < \infty \right)$$

- **Finite power signal or Power signal.** The total power of a signal is finite if

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \quad \left(P_{x_k} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |x_k|^2 < \infty \right)$$

- **Finite power Periodic signal or Power Periodic signal (T: period for continuous-time, N: period for discrete-time).**

$$P_x = \frac{1}{T} \int_{(T)} |x(t)|^2 dt < \infty \quad \left(P_{x_k} = \frac{1}{N} \sum_{(N)} |x_k|^2 < \infty \right)$$

  - The total power is not a norm except for periodic signals
  - An energy signal is a zero power signal
  - The energy of a non zero power signal is infinite

2. Spectral Density

If $X(t,\omega)$ is a realization of a stochastic $\{X(t,\omega), t \in \mathcal{T}\}$, its Fourier transform is given by

$$\mathcal{F}[X(t,\omega)] = \Phi_{X(t,\omega)}(f) = \int_{-\infty}^{\infty} X(t,\omega)e^{-j2\pi ft}dt$$

$\Phi_{X(t,\omega)}(f)$ is a random and to connect it with the autocorrelation function, we evaluate

$$\begin{aligned} E[\Phi_{X(t_1,\omega)}(f)\overline{\Phi_{X(t_2,\omega)}(f')}] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t_1,\omega)\overline{X(t_2,\omega)}]e^{-j2\pi ft_1}e^{j2\pi f't_2}dt_1\, dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1,t_2)e^{-j2\pi f(t_1-t_2)}e^{-j2\pi(f-f')t_2}dt_1\, dt_2 \end{aligned}$$

If the process is stationary, we have

$$E[\Phi_{X(t_1,\omega)}(f)\overline{\Phi_{X(t_2,\omega)}(f')}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1-t_2,0)e^{-j2\pi f(t_1-t_2)}e^{-2\pi(f-f')t_2}dt_1\, dt_2$$

2. Spectral Density

Denoting $\tau = t_1 - t_2$, introduce the bijective change of variable from $\mathbb{R}^2$ to $\mathbb{R}^2$

$$\begin{bmatrix} \tau \\ t_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \mathcal{G}(T)$$

Remark that $\det |\partial \mathcal{G}/\partial T| = 1$ and then we have

$$E[\Phi_{X(t_1,\omega)}(f)\overline{\Phi_{X(t_2,\omega)}(f')}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}e^{-2\pi(f-f')t_2}\,d\tau\,dt_2$$

leading to *(Cramer-Rao's Theorem)*

$$\boxed{\begin{gathered} E[\Phi_{X(t_1,\omega)}(f)\overline{\Phi_{X(t_2,\omega)}(f')}] = \underbrace{\left[\int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau\right]}_{\gamma_X(f)} \times \underbrace{\left[\int_{-\infty}^{\infty} e^{-2\pi(f-f')t_2}\ dt_2\right]}_{\delta(f-f')} \\ E[\,|\Phi_{X(t,\omega)}(f)|^2] = \gamma_X(f) \end{gathered}}$$

2. Spectral Density

Definition

Consider a stationary stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$. Its *power spectral density* $\gamma_X(f)$ is defined as the Fourier transform of the autocorrelation function. More precisely

- If the process is continuous-time

$$\boxed{\begin{array}{c} \gamma_X(f) = \displaystyle\int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau, \quad R_X(\tau) = \int_{-\infty}^{\infty} \gamma_X(f) e^{j2\pi f\tau} df \\ R_X(\tau) = E[X(t)\overline{X(t+\tau)}], \quad \gamma_X(f) = E[\,|\Phi_{X(t,\omega)}(f)|^2] \end{array}}$$

- If the process is discrete-time

$$\boxed{\begin{array}{c} \gamma_X(f) = \displaystyle\sum_{\tau=-\infty}^{\infty} R_X(\tau) e^{-j2\pi f t\tau}; \quad R_X(\tau) = \int_{-1/2}^{1/2} \gamma_X(f) e^{j2\pi f\tau} df \\ R_X(\tau) = E[X(t)\overline{X(t+\tau)}] \quad \gamma_X(f) = E[\,|\Phi_{X(t,\omega)}(f)|^2] \\ \text{The frequency range is } [-f_e/2, f_e/2] \text{ for a sampled signal at frequency } f_e \end{array}}$$

2. Spectral Density

PROPERTIES

- $\gamma_X(f) \geqslant 0$ for all f
- If the process is real, $\gamma_X(f) = \gamma_X(-f)$,
- $P_X = R_X(0) = m_X^2 + \sigma_X^2 = \int_{-\infty}^{\infty} \gamma_X(f)df$

A USEFUL INTEGRAL

- $\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = 1$
- $\mathcal{F}^{-1}[1] = \int_{-\infty}^{\infty} e^{j2\pi ft}df = \int_{-\infty}^{\infty} e^{-j2\pi ft}df = \delta(t) = \delta(-t)$
- $\int_{-\infty}^{\infty} e^{j2\pi ft}dt = \delta(f)$

3. Examples

EXAMPLE 1: telegraphic signal

For the telegraphic signal, stationary in a wide sense we have

$$\gamma_X(f) = \frac{1}{4}\delta(f) + \frac{1}{4}\frac{a}{a^2 + \pi^2 f^2}$$

Note that $P_X = R_X(0) = 1/2$

EXAMPLE 2

The process of example 2 is also stationary in the wide sense and

$$\gamma_X(f) = M^2\delta(f) + \frac{2A^2 b}{b^2 + 4\pi^2 f^2} \quad P_X = R_X(0) = A^2 + M^2$$

EXAMPLE 3

The process of example 3 is stationary in the wide sense and

$$\gamma_X(f) = \frac{1}{4}\sum_{i=0}^{N} A_i^2 p_i \left[\delta(f - f_0) + \delta(f + f_0)\right], \; P_X = R_X(0) = \frac{1}{2}\sum_{i=0}^{N} A_i^2 p_i$$

4. Filtering

- Many applications involve linear time-invariant (LTI) systems. It is important to know how the characteristics of random signals are modified by such systems.
- If $\{W(t,\omega), t \in \mathcal{T}\}$ is the input of a linear system, the output is a stochastic process defined by $\{Y(t,\omega), t \in \mathcal{T}\}$
- If the system is LTI, for an input realization, we obtain an output realization expressed as a convolution product

$$
\begin{aligned}
Y(t,\omega) &= h(t) \star W(t,\omega) = \int_{-\infty}^{\infty} h(u)W(t-u,\omega)du \\
&= \int_{-\infty}^{\infty} h(t-u)W(u,\omega)du
\end{aligned}
$$

where $h(u)$ is the impulse response of the LTI system.

4. Filtering

If the system is stable, i.e

$$\int_{-\infty}^{\infty} |h(u)| du < \infty$$

we have

$$\begin{aligned} E[Y(t)] &= E\left[\int_{-\infty}^{\infty} W(u,\omega) h(t-u) du\right] \\ &= \int_{-\infty}^{\infty} E[W(u)] h(t-u) du = \int_{-\infty}^{\infty} m_W(u) h(t-u) du \end{aligned}$$

- If W is at least first order stationary Y is first order stationary and we have

$$\boxed{E[Y(t)] = m_W \int_{-\infty}^{\infty} |h(v)| dv = H(0)\ m_W}$$

where $H(0)$ is the *static gain* of the LTI system

4. Filtering

$$
\begin{aligned}
R_Y(t_1,t_2) &= E[Y(t_1)\overline{Y(t_2)}] = E\left[\int_{-\infty}^{\infty} h(u)W(t_1-u,\omega)du \int_{-\infty}^{\infty} h(v)\overline{W(t_2-v,\omega)}dv\right] \\
&= \int_{-\infty}^{\infty} h(u)du \int_{-\infty}^{\infty} h(v)E[W(t_1-u)\overline{W(t_2-v)}]dv \\
&= \int_{-\infty}^{\infty} h(u)du \int_{-\infty}^{\infty} h(v)R_W(t_1-u,t_2-v)dv
\end{aligned}
$$

If W is stationary in wide sense, $R_W(t_1-u,t_2-v) = R_W(\tau+v-u)$, $\tau = t_1-t_2$, then Y is stationary in wide sense and

$$
\boxed{R_Y(\tau) = \int_{-\infty}^{\infty} h(u)du \int_{-\infty}^{\infty} h(v)R_W(\tau+v-u)dv}
$$

4. Filtering

The spectral density of the output is given by

$$\begin{aligned}\gamma_Y(f) &= \int_{-\infty}^{\infty} R_Y(\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} h(u) du \int_{-\infty}^{\infty} h(v) dv \int_{-\infty}^{\infty} R_W(\tau + v - u) e^{-j2\pi f\tau} d\tau\end{aligned}$$

But $e^{-j2\pi f\tau} = e^{-j2\pi f(\tau+v-u)} e^{j2\pi f v} e^{-j2\pi f u}$ and denoting $s = \tau + v - u$, we have

$$\begin{aligned}\gamma_Y(f) &= \int_{-\infty}^{\infty} h(u) e^{-j2\pi f u} du \int_{-\infty}^{\infty} h(v) e^{j2\pi f v} dv \int_{-\infty}^{\infty} R_W(s) e^{-j2\pi f s} ds \\ &= H(j2\pi f)\overline{H(j2\pi f)}\gamma_W(f)\end{aligned}$$

and then

$$\boxed{\gamma_Y(f) = |H(j2\pi f)|^2 \gamma_W(f)}$$