RANDOM SIGNALS

Chapter III

Random Signals

Objective of Chapter III

- Introduce a rigorous definition of random signals
- Present some important properties
- Introduce the main indicators characterizing such signals

1. Definition of a Stochastic Process

Definition

A stochastic process can be defined as a family of random variables depending explicitly on several parameters. In this course, we only consider one parameter belonging to a set T. This parameter will refer to the time

A FIRST EXAMPLE

Consider a stochastic process consisting of a family of functions defined by

$$A\cos(2\pi f t + \phi)$$

where the values of A, f and ϕ are obtained rolling three dies, i.e. six values are possible for A, f and ϕ. Each possible function is called a *realization* of the stochastic process.

1. Definition of a Stochastic Process

Definition

Let a random experiment characterized by $(\Omega, \mathcal{P}(\Omega), P)$. A stochastic process $\{X(t, \omega), t \in \mathcal{T}\}$ is defined by

$$\begin{array}{ccc} \mathcal{T} \times \Omega & \to & \mathcal{X} \\ (t, \omega) & \to & X(t, \omega) \end{array}$$

We can see a stochastic process as

- *A family of functions of* t *and* ω
- *A simple function of* t *when* ω *is fixed*
- *A random variable when* t *is fixed*
- *A number when* t *and* ω *are fixed*

A FIRST EXAMPLE (continued)

When ω is fixed, we have a realization which is a periodic function of time among the $6 \times 6 \times 6 = 216$ possible ones. When t is fixed, we have a simple random variable taking $6^3 = 216$ values and when t and ω are fixed, we have a simple real.

1. Definition of a Stochastic Process

- If the time is discrete, $\mathcal{T}$ is countable and we say that the process is a *discrete-time stochastic process*
- If time is continuous, the conditions of randomness are active at each instant, we say that the process is a *continuous-time stochastic process*
- If the stochastic process takes values in a continuous set (for example $\mathcal{X} = \mathbb{R}$ or $\mathbb{C}$), we say that the process is a *continuous-state stochastic process*
- If the stochastic process takes values in a discrete set (for example $\mathcal{X} = \mathbb{Z}$), we say that the process is a *discrete-state stochastic process*
- A stochastic process is completely characterized if for all k, the joint density functions

$$f_{X_1 X_2 \dots X_k}(x_1, x_2, \dots, x_k) \text{ where } x_i = x(t_i, \omega), i = 1, \dots, k$$

are known. In general, they are unknown and only partial descriptions are available through some specific moments.

2. Characteristics of order 1

- When t is fixed, say t_1, $X(t_1, \omega) \triangleq X_1$ is a random variable. Its distribution is given by

$$F_{X_1}(x_1, t_1) = P(X_1 \leqslant x_1)$$

and its density by

$$f_{X_1}(x_1, t_1) = \frac{\partial F_{X_1}(x_1, t_1)}{\partial x_1}$$

- It is possible to deduce its expectation

$$\boxed{m_X(t_1) = E[X_1] = \int_{\mathcal{X}} x_1 f_{X_1}(x_1, t_1) dx_1}$$

More generally if $g(\bullet)$ is a given function, we have

$$E[g(X_1)] = \int_{\mathcal{X}} g(x_1) f_{X_1}(x_1, t_1) dx_1$$

- When $g(X_1) = (X_1 - m_X(t_1))\overline{(X_1 - m_X(t_1)}$, the moment is the variance $\sigma_X^2(t_1)$.

2. Characteristics of order 1

- As pointed out, the complete characterization of the stochastic process needs the knowledge of all the distributions functions when time is fixed at one, two, three . . . instants.
- In general, from a practical point of view, the knowledge of all joint distributions is impossible.
- The characteristics of order 1 do not consider the time evolution of the process because time is fixed. To have an information about the time evolution, at least two instants have to be considered. In that case, characteristics of order 2 are involved.
- Concerning electronics or control systems, characteristics of order 2 will be sufficient to obtain a spectral characterization of the stochastic processes of interest.

3. Characteristics of order 2

- When t is fixed at two instants, say t_1 and t_2, $X(t_1,\omega) \triangleq X_1$ and $X(t_2,\omega) \triangleq X_2$ are jointly distributed random variables whose joint distribution is given by

$$F_{X_1X_2}(x_1,x_2,t_1,t_2) = P(X_1 \leqslant x_1, X_2 \leqslant x_2)$$

and the joint density by

$$f_{X_1X_2}(x_1,x_2,t_1,t_2) = \frac{\partial^2 F_{X_1X_2}(x_1,x_2,t_1,t_2)}{\partial x_1 \partial x_2}$$

- We can define the *autocorrelation function*

$$\boxed{R_X(t_1,t_2) = E[X_1\overline{X_2}] = \int_{\mathcal{X}}\int_{\mathcal{X}} x_1\overline{x_2} f_{X_1X_2}(x_1,x_2,t_1,t_2)dx_1dx_2}$$

- The *autocovariance function* is defined by

$$\boxed{C_X(t_1,t_2) = E[(X_1 - m_X(t_1))(\overline{X_2 - m_X(t_2)})]}$$

3. Characteristics of order 2

- Recall that $E[X\overline{Y}]$ is a product scalar. In that context, the autocorrelation is a measure of the correlation between X and Y, in some sense a measure of the "resemblance" between X and Y.
- If $E[X\overline{Y}]$ is significative, the correlation between X and Y is significative characterizing intuitively a "flat" signal
- A "small value" of $E[X\overline{Y}]$ is characteristic of a signal with fluctuations.
- There is a relation between $E[X\overline{Y}]$ and the variability of the stochastic process and then a relation with its frequency content
- Note that $E[X\overline{Y}]$ contains an information about the average. In the autocovariance function, only the fluctuations around the average value are involved
- We have the following properties

$$\boxed{\begin{array}{c} R_X(t_1,t_2) = \overline{R_X(t_2,t_1)} \\ R_X(t_1,t_1) = E[|X(t_1)|^2] \ (\textit{Instantaneous Power}) \\ |R_X(t_1,t_2)|^2 \leqslant R_X(t_1,t_1)R_X(t_2,t_2) \ (\textit{Schwarz Inequality}) \\ C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)\overline{m_X(t_2)} \end{array}}$$

4. Intercorrelation function

Definition

Let two stochastic processes $\{X(t, \omega), t \in \mathcal{T}\}$ and $\{Y(t, \omega), t \in \mathcal{T}\}$. We define the intercorrelation function between X and Y by

$$R_{XY}(t_1, t_2) = E[X(t_1)\overline{Y(t_2)}] = \iint x_1 y_2 f_{XY}(x_1, y_2, t_1, t_2) dx_1 dy_2$$

where $f_{XY}(x_1, x_2, t_1, t_2)$ is the joint probability density of $X(t_1)$ and $Y(t_2)$. By extension, the intercovariance function is defined by

$$C_{XY}(t_1, t_2) = E[(X_1 - m_X(t_1))(\overline{Y_2 - m_Y(t_2)})]$$

These indicators measure the correlation between the realizations of two different stochastic processes. We have

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - m_X(t_1)\overline{m_Y(t_2)}$$

5. Examples

EXAMPLE 1: telegraphic signal

Consider a stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$ whose realizations are signals taking values in $\mathcal{X} = \{0, 1\}$.

- The instants where the value of the signal $X(t,\omega)$ changes are random and follow a Poisson's distribution. More precisely, the probability that the value of the signal $X(t,\omega)$ changes k times over a time interval of length T is given by

$$P(k, T) = \frac{(aT)^k}{k!} e^{-aT}$$

  where a is the average number of changes per unit of time.
- At the origin, the realizations take values 0 or 1 with a probability $1/2$.

5. Examples

Consider a stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$ whose realizations are signals taking values in $\mathcal{X} = \{0, 1\}$.

- The average value is given by

$$m_X(t) = E[X(t)] = \sum_{i=1}^{2} x_i P(X(t) = x_i) = 0 \times P(X(t) = 0) + 1 \times P(X(t) = 1) = P(X(t) = 1)$$

- Introduce

$$\mathrm{Even}([t_1, t_2]) \triangleq \text{even number of changes over interval } [t_1, t_2]$$
$$\mathrm{Odd}([t_1, t_2]) \triangleq \text{odd number of changes over interval } [t_1, t_2]$$

$$\begin{aligned}
P(X(t) = 1) &= P(X(0) = 0 \text{ and } \mathrm{Odd}([0,t])) + P(X(0) = 1 \text{ and } \mathrm{Even}([0,t])) \\
&= P(X(0) = 0) \times P(\mathrm{Odd}([0,t]) | X(0) = 0) + P(X(0) = 1) \times P(\mathrm{Even}([0,t]) | X(0) = 1) \\
&= P(X(0) = 0) \times P(\mathrm{Odd}([0,t])) + P(X(0) = 1) \times P(\mathrm{Even}([0,t])) \\
&= 1/2 \times [P(\mathrm{Odd}([0,t])) + P(\mathrm{Even}([0,t]))] = 1/2
\end{aligned}$$

The time independence of $m_X(t)$ is due to the equiprobability of the values taken by the realizations at initial time $t = 0$

5. Examples

The autocorrelation function is given by

$$
\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)\overline{X(t_2)}] = \sum_{i=1}^{2}\sum_{j=1}^{2} x_i x_j P(X(t_1)=x_i \text{ and } P(X(t_2)=x_j)) \\
&= 1 \times 1 \times P((X(t_1)=1) \text{ and } P(X(t_2)=1)) \\
&= 1/2 \times P(X(t_2)=1|X(t_1)=1)
\end{aligned}
$$

$$
\boxed{R_X(t_1,t_2) = \frac{1+e^{-2a|t_2-t_1|}}{4}}
$$

because we have

$$
\begin{aligned}
P(X(t_2)=1|X(t_1)=1) &= P(Even([t_1,t_2])) \\
&= e^{-a|t_2-t_1|}\sum_{j=0}^{\infty}\frac{a|t_2-t_1|^{2j}}{(2j!} = \frac{1+e^{-2a|t_2-t_1|}}{2}
\end{aligned}
$$

$$
\left(\left\{\begin{array}{l}
e^x = \displaystyle\sum_{i=0}^{\infty}\frac{x^i}{i!} = \sum_{j=0}^{\infty}\frac{x^{2j}}{(2j)!} + \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!} \\
e^{-x} = \displaystyle\sum_{j=0}^{\infty}\frac{x^{2j}}{(2j)!} - \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}
\end{array}\right. \Rightarrow \sum_{j=0}^{\infty}\frac{x^{2j}}{(2j)!} = \frac{e^x+e^{-x}}{2}\right)
$$

EXAMPLE 2

- Consider a stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$ whose realizations are constituted by piecewise constant signals over random intervals and whose amplitude changes randomly. The probability to have a change in amplitude during the interval $[t_1, t_2]$ is given by

$$P([t_1, t_2]) = 1 - e^{-b|t_2 - t_1|}$$

- The amplitude is a random variable whose average is M and variance is A^2.The amplitudes before and after an instant of change are independent.
- Note that the set $\mathcal{X}$ can be a discrete or a continuous set.
  In the first case, *the process is a discrete-state, continuous-time stochastic process*.
  In the second one, *the process is a continuous-state, continuous-time stochastic process*. The only needed informations about the amplitude distribution are the variance and average.

5. Examples

EXAMPLE 2

- We have

$$\boxed{m_X(t) = E[X(t)] = M}$$

- For the autocorrelation function, note that

$$R_X(t_1, t_2) = E[X(t_1)\overline{X(t_2)}] = \begin{cases} E[X(t)^2] & \text{if} \quad X(t_1) = X(t_2) \\ E[X(t_1)]E[X(t_2)] & \text{if} \quad X(t_1) \neq X(t_2) \end{cases}$$

Then

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t)^2].P(x(t_1) = x(t_2)) + E[X(t_1)]E[X(t_2)].P(x(t_1) \neq x(t_2)) \\ &= (A^2 + M^2).[1 - P(x(t_1) \neq x(t_2))] + M^2.P(x(t_1) \neq x(t_2)) \\ &= A^2 + M^2 - A^2[1 - e^{-b|t_2 - t_1|}] \\ &= \boxed{M^2 + A^2 e^{-b|t_2 - t_1|}} \end{aligned}$$

5. Examples

EXAMPLE 3

- Consider a stochastic process $\{X(n,\omega), n \in \mathcal{T}\}$ whose realizations are constituted by signals expressed by

$$X(n,\omega) = A(\omega)\sin(2\pi f_0 n + \phi(\omega))$$

- ϕ is uniformly distributed over the interval $[0, 2\pi]$.
- $A \in \mathcal{A} = \{A_1, A_2, A_3, \ldots, A_N\}$ with respectively probabilities $\{p_1, p_2, \ldots, p_N\}$
- The random variables A and ϕ are independent
- This stochastic process is discrete-time.

5. Examples

EXAMPLE 3

- We have

$$\begin{aligned} m_X(t) &= E[X(n)] = E[A.\sin(2\pi f_0 n + \phi)] \\ &= E[A] \times E[\sin(2\pi f_0 n + \phi)] \quad \big(\text{because of independence of } A \text{ and } \phi\big) \\ &= \boxed{\sum_{i=0}^{N} A_i p_i \times \int_0^{2\pi} \sin(2\pi f_0 n + \phi) f_\phi(\phi) d\phi = 0} \quad \left(f_\phi(\phi) = \frac{1}{2\pi}, \; 0 \leqslant \phi \leqslant 2\pi \right) \end{aligned}$$

- For the autocorrelation function, note that

$$\begin{aligned} R_X(n_1, n_2) &= E[X(n_1)\overline{X(n_2)}] \\ &= E[A \sin(2\pi f_0 n_1 + \phi) \times A \sin(2\pi f_0 n_2 + \phi)] \\ &= E[A^2] \times E[\tfrac{1}{2}\cos(2\pi f_0 (n_1 - n_2)) - \tfrac{1}{2}\cos(2\pi f_0 (n_1 + n_2) + 2\phi)] \\ &\quad \big(\text{by independence of } A \text{ and } \phi, \text{ and } 2\sin a \sin b = \cos(a-b) - \cos(a+b)\big) \\ &= \boxed{\frac{1}{2}\sum_{i=0}^{N} A_i^2 p_i \cos(2\pi f_0 (n_1 - n_2))} \end{aligned}$$

6. Stationarity

Definition

A stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$ is *stationary* or *strictly (strongly) stationary* if the unconditional joint probability density does not change when shifted in time, i.e.

$$f_{X_{t_1} X_{t_2} \dots X_{t_k}} = f_{X_{t_1-\tau} X_{t_2-\tau} \dots X_{t_k-\tau}} \ \forall t_1, t_2, \dots, t_k, \tau \in \mathcal{T} \text{ and } k \in \mathbb{N}$$

The strict stationarity is a very strong concept and is really difficult to verify in practice. A weaker form of stationarity is often used in signal processing

Definition

A stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$ is *weakly stationary* or *wide-sense stationary* if the characteristics of order 1 and order 2 are independent of the origin of time, i.e.

1. $m_X(t) = m_X(t-\tau) \ \ \forall \tau \in \mathcal{T}$
2. $R_X(t_1, t_2) = R_X(t_1 - t_2, 0) \triangleq R_X(\tau) \ \ \forall t_1, t_2 \in \mathcal{T}, \ \tau = t_1 - t_2$
3. $R_X(0) = E[|X(t)|^2] < \infty \ \ \forall t \in \mathcal{T}$

5. Examples

EXAMPLE 1: telegraphic signal

The telegraphic signal is stationary in a wide sense because $m_X(t) = 1/2$ and

$$R_X(t_1, t_2) = R_X(\tau) = \frac{1 + e^{-2a|\tau|}}{4}, \quad \tau = t_2 - t_1$$

Note that $R_X(\tau) \leqslant R_X(0) = E[|X(t)|^2] = 1/2$ instantaneous power.

EXAMPLE 2

The process of example 2 is also stationary in the wide sense, $m_X(t) = M$ and $R_X(\tau) = M^2 + A^2 e^{-b|\tau|}$, $R_X(0) = E[|X(t)|^2] = A^2 + M^2$.

EXAMPLE 3

The process of example 3 is stationary in the wide sense, $m_X(t) = 0$ and

$$R_X(\tau) = \frac{1}{2} \sum_{i=0}^{N} A_i^2 p_i \cos(2\pi f_0 \tau), \; R_X(0) = E[|X(t)|^2] = \frac{1}{2} \sum_{i=0}^{N} A_i^2 p_i$$

7. Ergodicity

Definition

A stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$ is *ergodic* if its statistical properties can be deduced from a single, sufficiently long, random sample of the process.

A stochastic process is *ergodic* if time average characteristics are deterministic

Strict ergodicity is a very strong concept difficult to verify in practice. A weaker concept of ergodicity can be defined as follows.

Definition

A stochastic process $\{X(t,\omega), t \in \mathcal{T}\}$ is called *ergodic in the wide sense* if

1. $< X(t,\omega) >= \lim_{t\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t,\omega)dt = E[X(t)] = m_X$
2. $< X(t,\omega)\overline{X(t-\tau,\omega)} >= \lim_{t\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t,\omega)\overline{X(t-\tau,\omega)}dt =$
   $\lim_{t\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t,t-\tau)dt = R_X(\tau)$ (if the process is weak stationary)

7. Ergodicity (discrete-time)

For discrete-time processes, the average characteristic of order 1 is given by

$$< X(t,\omega) >= \lim_{t\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} X(t,\omega)$$

and characteristic of order 2 by

$$< X(t,\omega)\overline{X(t-\tau,\omega)} >= \lim_{t\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} X(t,\omega)\overline{X(t-\tau,\omega)}$$

- *The original definition of ergodicity only concerned the first moment. First order ergodicity implies first order stationarity. The above definition of ergodicity is an extended version to moments of higher orders.*

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7. Ergodicity

EXAMPLES

Consider two random variables A with $E[A] = a$ and ϕ uniformly distributed over $[0, 2\pi]$

- Consider a stochastic process whose realizations are $X(t, \omega) = Y(t) + A$. The process is not stationary because $E[X(t)] = Y(t) + a$ is time dependent and not ergodic because $< X(t, \omega) >=< Y(t) > +A$ is random.
- If $Y(t) = y_0$, the process is first order stationary because $E[X(t)] = y_0 + a$ is time independent and remains non ergodic because $< X(t, \omega) >= y_0 + A$ is random.

7. Ergodicity

EXAMPLES (continued)

- Consider realizations of a stochastic process, $X(t,\omega) = A\cos(2\pi ft)$ where $E[A] = 0$. The process is stationary in a wide sense, but not ergodic in the wide sense because $< X(t,\omega)X(t-\tau,\omega) > \neq R_X(\tau)$. In fact

$$< X(t,\omega)X(t-\tau,\omega) >= \frac{1}{T}\int_{(T)} X(t,\omega)X(t-\tau,\omega)dt = \frac{A^2}{2}\cos(2\pi f\tau)$$

  is random and $R_X(\tau) = \dfrac{E[A^2]}{2}\cos(2\pi f\tau)$. However the process is first order ergodic because $E[X(t)] =< X(t,\omega) >= 0$.

- The stochastic process whose realizations are defined by $X(t,\omega) = \cos(2\pi ft + \phi)$ is stationary and ergodic in a wide sense.