RANDOM SIGNALS: (SA - I3MITS11 - I3MAEL21)

- **LECTURE**: *5 Sessions (6h15)* - Germain Garcia
- **TUTORIALS**: *4 Sessions (5h00)* - Germain Garcia, Alexandre Boyer
- **LAB WORK**: *1 Session (2h45)* - Germain Garcia, Nadim Nasreddine. (Only for 3IMACS)
- **EXAMS**: *written Exam (duration: about 1h15): questions about lecture topics, tutorials and course application*
- **PREREQUISITES**: deterministic signals (Fourier Series and Fourier Transform), Course in analysis.
- **MOODLE** : *these slides and complementary documents associated with tutorials can be downloaded from the MOODLE platform*

RANDOM SIGNALS

Chapter I

Introduction

Objective of Chapter I

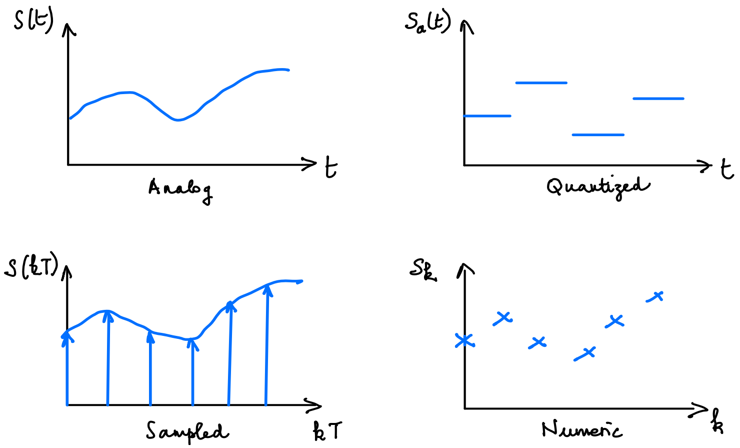
- Recall some basic facts concerning deterministic signals
- Present the main lines of the course
- Give a list of references

1. Classification of deterministic signals

- A deterministic signal is completely defined by one or more independent variables or factors *(for example time)*
- A deterministic signal remains invariant when remeasured over equal ranges of the factors
- We can classify the signals *(morphological classification)*

- **Analog Signal**. Continuous time and continuous amplitude signal : function of a continuous independent variable, time ($\mathbb{R}$). The range of the amplitude of the function is also continuous ($\mathbb{R}$ or $\mathbb{C}$): can be represented as a function "$s(t)$"
- **Quantized Signal**. Continuous time and discrete amplitude signals are a function of a continuous independent variable, time ($\mathbb{R}$) - but the amplitude is discrete (discrete subset $A = \{a_1, a_2, \cdots\} \subset \mathbb{R}$ or $\mathbb{C}$): can be represented as a function "$s_a(t)$"
- **Sampled Signal**. Discrete time and continuous amplitude signals are functions of a quantized or discrete independent time variable ($\mathbb{Z}$), while the range of amplitudes is continuous($\mathbb{R}$ or $\mathbb{C}$): can be represented as a function "$s(kT)$, $T > 0$ *sampling period* "
- **Numeric Signal** Discrete time and discrete amplitude signals are functions where both the independent time variable ($\mathbb{Z}$) and the amplitude are discrete (discrete subset $A = \{a_1, a_2, \cdots\} \subset \mathbb{R}$ or $\mathbb{C}$): can be represented as a series "s_k"

1. Classification of deterministic signals



Morphological Classification

1. Classification of deterministic signals

We can classify the signals from an energy point of view

- **Finite energy signal or Energy signal (L_2).** The energy of a signal $s(t)$ (s_k) is finite if

$$E_s = \|s\|_2^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt < \infty \quad \left(E_{s_k} = \sum_{k=-\infty}^{\infty} |s_k|^2 < \infty \right)$$

It is important to note that $E_s^{1/2}$ ($E_{s_k}^{1/2}$) is a norm induced by the scalar product

$$< s_1, s_2 >= \int_{-\infty}^{\infty} s_1(t) s_2^*(t) dt \quad \left(< s_1, s_2 >= \sum_{k=-\infty}^{\infty} s_{1k} s_{2k}{}^* \right)$$

- **Finite power signal or Power signal.** The total power of a signal is finite if

$$P_s = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{1}{T}}^{\frac{1}{T}} |s(t)|^2 dt < \infty \quad \left(P_{s_k} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |s_k|^2 < \infty \right)$$

  - The total power is not a norm except for periodic signals
  - An energy signal is a zero power signal
  - The energy of a non zero power signal is infinite

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE LAAS CNRS

2. Signal Transformations

- To make easier analysis and simplify complicated operations, the idea is to transform signals
- In that context the notion of frequency is central and both time domain and frequency space are whole and consistant ways of looking at a signal
- From a mathematical point of view, it is important to know the limits of such transformations
- The notion of frequency is at first level associated to periodicity and to periodic signals

A signal $s(t)$ (s_k) is periodic if there exists a positive number $T \in \mathbb{R}$ $(T \in \mathbb{N})$ such that

$$\forall t \in \mathbb{R}\ (k \in \mathbb{Z})\ \ \forall l \in \mathbb{Z}, s(t) = s(t + lT)\ \ (s_k = s_{k+lT})$$

2. Fourier Series

PERIODIC SIGNALS (Fourier Series)

If the signal $s(t)$ is periodic of period T and satisfies (*power signal,* L^2_T *periodic signals*)

$$\frac{1}{T}\int_t^{t+T} |s(t)|^2 dt < \infty$$

then we have (almost everywhere)

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi f_0 t}, \quad c_n = \frac{1}{T}\int_t^{t+T} s(u)e^{-jn2\pi f_0 u} du, \quad f_0 = \frac{1}{T}$$

Defining the scalar product (dot product)

$$< s_1, s_2 >= \frac{1}{T}\int_t^{t+T} s_1(u)s_2^*(u)du$$

$\{e_n = e^{jn2\pi f_0 t} : n = 1, \cdots, \infty\}$ is an orthonormal basis (Show it) and the series can be expressed as

$$s(t) = \sum_{n=-\infty}^{\infty} < s, e_n > e_n, \quad c_n =< s, e_n >$$

2. Fourier Series

- The equality between the series and the function $s(t)$ has to be understood in the following sense

$$\lim_{N\to\infty} \|S_N - s)\|^2 = 0$$

where

$$S_N(t) = \sum_{n=-N}^{N} c_n e^{jn2\pi f_0 t},$$

- For some values of t, the equality between s(t) and the series is not ensured and depends of the properties of $s(t)$.

(Dirichlet Theorem) : If $s(t)$ is a piecewise differentiable function we have

$$\frac{s(t^-) + s(t^+)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi f_0 t}$$

2. Fourier Series

Theorem (Parseval equality)

$$P_s = \frac{1}{T}\int_t^{t+T} |s(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Definition

If the periodic signal $s(t)$ is decomposable in Fourier series, its power spectral density $\Phi_s(f)$ is defined by

$$\Phi_s(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

and its phase spectrum by

$$\varphi_s(f) = \sum_{n=-\infty}^{\infty} \arg[c_n]\, \delta(f - nf_0)$$

We have

$$\int_{-\infty}^{\infty} \Phi_s(f) df = \sum_{n=-\infty}^{\infty} |c_n|^2 = P_s$$

2. Fourier Series

We can deduce alternate expressions for the series (Show it)

$$\begin{aligned} s(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n2\pi f_0 t) + b_n \sin(n2\pi f_0 t) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin(n2\pi f_0 t + \Psi_n) \end{aligned}$$

$$a_0 = \frac{1}{T}\int_t^{t+T} s(u)du$$

$$a_n = \frac{2}{T}\int_t^{t+T} s(u)\cos(n2\pi f_0 u)du, \;\; b_n = \frac{2}{T}\int_t^{t+T} s(u)\sin(n2\pi f_0 u)du, \; n > 1$$

$$A_n = \sqrt{a_n^2 + b_n^2}, \;\; \Psi_n = \arctan(a_n/b_n)$$

$$c_n = \frac{a_n - jb_n}{2}, \;\; a_n = 2\,\mathfrak{Re}[c_n], \;\; b_n = -2\,\mathfrak{Im}[c_n]$$

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE LAAS CNRS

2. Fourier Series: properties

Consider periodic signals $s(t)$, $s_1(t)$ and $s_2(t)$ whose Fourier coefficients are respectively c_n, c_{1n} and c_{2n}. We have the following properties

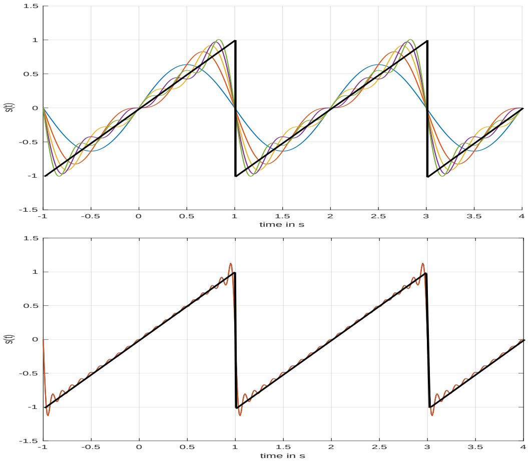
- *Linearity:* Fourier coefficients of $\alpha.s_1 + \beta.s_2$ are $\alpha.c_{1n} + \beta.c_{2n}$
- *Time reversal:* Fourier coefficients of $s(-t)$ are c_{-n}
- *Time shift:* Fourier coefficients of $s(t+a)$ are $e^{-jnf_0 a}\, c_n$
- *Derivation:* Fourier coefficients of $s^{(k)}(t)$ are $(jn2\pi f_0)^k\, c_n$
- *Conjugation:* Fourier coefficients of $s^*(t)$ are ${c_{-n}}^*$. This last property implies that the spectrum of a real signal is symmetric with respect to the frequency 0.

2. Fourier Series

EXAMPLE

For a periodical ramp signal
$(A = 1,\ T = 2)$

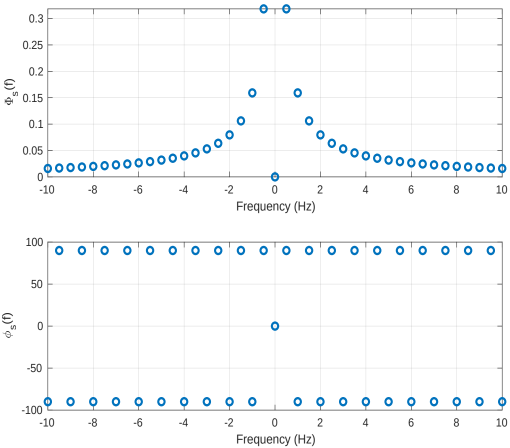
- $s(t)$ is odd, then
  $s(t) = -s(-t) \Rightarrow \forall\ n,\ a_n = 0$
- $b_n = (-1)^{n+1}\dfrac{2A}{n\pi}$
- $\dfrac{1}{2}\displaystyle\int_{-1}^{1} |s(t)|^2 dt = \dfrac{1}{3} \approx 0.3333$
- $\displaystyle\sum_{n=-20}^{20} |c_n|^2 \approx 0.3235$



Fourier Series - top: 5 harmonics - bottom: 20 harmonics

2. Fourier Series

EXAMPLE: Spectra



Power spectral density (top) and phase spectrum (bottom)

3. Fourier Transform

- The idea is to extend the Fourier series for non periodic signals. For doing that intuitively, introduce

$$f_n = \frac{n}{T} \Rightarrow \Delta f = f_n - f_{n-1} = \frac{1}{T}$$

We have

$$s(t) = \sum_{n=\infty}^{\infty} T.c_n \; e^{j2\pi n\Delta f.t}\Delta f, \;\; T.c_n = \int_{-\infty}^{\infty} s(t)\; e^{-j2\pi n\Delta f.t} dt \triangleq S(n\Delta f)$$

Non periodic signal can be interpreted as a periodic signal of infinite period, then $T \to \infty$, and $\Delta f \to df$, $n\Delta f \to f$.

Under some mathematical conditions depending of the signal $s(t)$, we could have

$$s(t) = \sum_{n=\infty}^{\infty} S(n\Delta f)\; e^{j2\pi n\Delta f.t} \underset{T\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} S(f)\; e^{2\pi f.t} df$$

$$T.c_n = S(n\Delta f) \underset{T\to\infty}{\longrightarrow} S(f) = \int_{-\infty}^{\infty} s(t)\; e^{-j2\pi f.t} dt$$

$S(f)$ is the Fourier transform.

3. Fourier Transform

NON PERIODIC SIGNALS (Fourier Transform)

If the signal $s(t)$ satisfies (*energy signal*)

$$\int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$$

then we have (almost everywhere)

$$s(t) = \int_{-\infty}^{\infty} S(f)\, e^{2\pi f.t} df, \quad S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi f.t} dt$$

Defining the scalar product (dot product)

$$< s_1, s_2 >= \int_{-\infty}^{\infty} s_1(t) s_2^*(t) dt$$

$\{e_f = e^{j2\pi f t} : f \in \mathbb{R}\}$ is an orthonormal basis (Show it) and (almost everywhere)

$$s(t) = \int_{-\infty}^{\infty} < s, e_f > e_f\, df, \quad S(f) =< s, e_f >$$

3. Fourier Transform

Theorem (Parseval equality)

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |S(f)|^2 df$$

Definition

If the signal $s(t)$ is an energy signal, its energy spectral density $\Phi_s(f)$ is defined by

$$\Phi_s(f) = |S(f)|^2$$

and its phase spectrum by

$$\varphi_s(f) = \arg[S(f)]$$

We have

$$\int_{-\infty}^{\infty} \Phi_s(f) df = \int_{-\infty}^{\infty} |s(t)|^2 dt = E_s$$

3. Fourier Transform: properties

Consider energy signals $s(t)$, $s_1(t)$ and $s_2(t)$ and denote the Fourier transform $\mathcal{F}[\bullet]$. We have the following properties

- *Linearity:* Fourier coefficients of $\mathcal{F}[\alpha.s_1 + \beta.s_2]$ are $\alpha.\mathcal{F}(s_1) + \beta.\mathcal{F}[s_2]$
- *Conjugation:* $\mathcal{F}[s^*](f) = \mathcal{F}[s](-f)^*$
- *Time shift:* If $g(t) = s(t + a)$, $\mathcal{F}[g](f) = e^{j2\pi a f}.\mathcal{F}[s](f)$
- *Derivation:* Fourier coefficients of $\mathcal{F}[s^{(k)}](f) = (j2\pi f)^k \mathcal{F}[s](f)$
- *Scaling:* If $g(t) = s(t/a)$, $\mathcal{F}[g](f) = |a|.\mathcal{F}[s](af)$
- *Convolution:* $\mathcal{F}\left[(s_1 \star s_2)(t) = \int_{-\infty}^{\infty} s_1(t)s_2(t-\tau)d\tau\right] = \mathcal{F}[s_1].\mathcal{F}[s_2]$

3. Fourier Transform: discrete time signal

If the discrete time signal s_k satisfies (*energy signal*)

$$\sum_{k=-\infty}^{\infty} |s_k|^2 < \infty$$

then we have

$$s_k = \int_{-1/2}^{1/2} S(f)\, e^{j2\pi f.k} df, \quad S(f) = \sum_{k=-\infty}^{\infty} s_k\, e^{-j2\pi f.k}$$

Note that $S(f)$ is periodic of period 1.

If s_k are the samples of a sampled signal i.e. $s_k = s(kT_e)$ where T_e is the sampling period and $f_e = 1/T_e$ the sampling frequency, the previous formulas become

$$s(kT_e) = \int_{-f_e/2}^{f_e/2} S(f)\, e^{2\pi f k T_e} df, \quad S(f) = \sum_{k=-\infty}^{\infty} s(kT_e)\, e^{-j2\pi f k T_e}$$

3. Fourier Transform: discrete time signal

Theorem (Parseval equality)

$$E_{s_k} = \sum_{-\infty}^{\infty} |s_k|^2 = \int_{-1/2}^{1/2} |S(f)|^2 df$$

Definition

If the signal s_k is an energy signal, its energy spectral density $\Phi_{s_k}(f)$ is defined by

$$\Phi_{s_k}(f) = |S(f)|^2$$

and its phase spectrum by

$$\varphi_{s_k}(f) = \arg[S(f)]$$

We have

$$\int_{-1/2}^{1/2} \Phi_s(f)df = \sum_{-\infty}^{\infty} |s_k|^2 = E_{s_k}$$

The range of frequencies is $[-1/2, 1/2]$, ($[-f_e/2, f_e/2]$ for a sampled signal).

4. Random signals

- The idea in this course is to extend the previous analysis to the case of random signals
- But what is a random signal ?

Definition

A *random* or a *stochastic* signal is a signal which is not reproduced identically in an experiment involving a priori the same experimental conditions

- In fact behind such an experiment, we can obtain a set of possible random signals. This set is called a *random* or a *stochastic process*
- A random signal is also called a *realization* of a stochastic process
- In that context, the associated models are probabilistic models and a stochastic process is characterized through *statistical descriptors*

5. Outline of the course

- Chapter I - Introduction
- Chapter II - Probability Theory and Random Variables (reminders)
- Chapter III - Random Signals
- Chapter IV - Spectral Analysis of Random Signals

6. References

- Gérard Scorletti. *Traitement du Signal*. Ecole Centrale de Lyon. https://cel.archives-ouvertes.fr/cel-00673929v3 . *covers the deterministic and random signals with interesting examples.*
- Matthieu Kowalski. *Traitement du Signal*. Université Paris-Saclay. http://hebergement.u-psud.fr/mkowalski/teaching.html. *Another interesting reference more complete, contains in particular wavelet transform.*
- J. L. Doob. *Stochastic Processes*. John Willey & Sons, Inc. London. 1964. *A very classical reference in the domain, but difficult.*

6. References

- A. H. Jazwinski. *Stochastic Processes and Filtering Theory*. Mathematics in Sciences and Engineering. Vol.64. Academic Press. New-York and London. 1970. *Another classical reference in the domain with a very good introduction to Kalman Filtering. Difficult.*
- W. Appel. *Mathématiques pour la physique, 5ième Edition*. Editions H & K. Paris . 2017. *An excellent book on all the needed mathematics for the models in physics. Rigorous and widely illustrated.*
- B. Beauzamy. *Méthodes probabilistes pour l'étude des phénomènes réels.* Société de Calcul Mathématique S.A., 2004. *An excellent book giving the minimal concepts of probability theory that can be efficiently used in the context of real situations.*