

# CONTROL SYSTEMS DESIGN

## Chapter III

### Responses of LTI Systems

## Objective of Chapter III

- When a model is available, analyze the behavior of the system for a given input
- Evaluate the dynamical behavior of the system through time constants, poles and other indicators extracted from a model
- Evaluate the dynamical behavior of the system in the frequency domain (frequency response)
- Have a good knowledge of first and second order systems and extrapolate this knowledge to systems of higher orders

# 1. From differential equation

Given a set of initial conditions and given an input  $u(t)$ , the response is obtained by the following three steps

1. Solve the homogeneous equation (without the right member)

$$a_n \frac{d^{(n)}y(t)}{dt^n} + a_{n-1} \frac{d^{(n-1)}y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

Find the characteristic roots, solutions of

$$D(r) = a_0 + a_1 r + \dots + a_n r^n$$

- If the root  $r_i$  is single, its contribution to the solution is  $A_i e^{r_i t}$ ,
- If the root  $r_i$  is multiple with multiplicity  $l$ , its contribution is  $(A_1 + A_2 t + \dots + A_l t^l) e^{r_i t}$ .

# 1. From differential equation

2. The solution of homogeneous equation is given by

$$y_1(t) = \sum_{i=1}^r P_i(t)e^{r_i t}$$

where  $P_i(t)$  are polynomials of order  $l_i$  where  $l_i$  are the multiplicity orders of characteristic roots  $r_i$ ,  $\sum_{i=1}^r l_i = n$  (order of the system)

3. Find a particular solution  $y_2(t)$  of the complete differential equation (with the right member), i.e. to

$$\begin{aligned} a_n \frac{d^{(n)}y(t)}{dt^n} + a_{n-1} \frac{d^{(n-1)}y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_m \frac{d^{(m)}u(t)}{dt^m} + b_{m-1} \frac{d^{(m-1)}u(t)}{dt^{m-1}} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned}$$

# 1. From differential equation

4. The complete solution is obtained as

$$y(t) = \sum_{i=1}^r P_i(t)e^{r_i t} + y_2(t)$$

The constants are determined using initial conditions.

An alternative for step 3. is to use the technique of the variation of constants.

# 1. From differential equation

## EXAMPLE

Let the system described by

$$y'' + 3y' + 2y = u + u', \quad y(0) = 1, \quad y'(0) = -1$$
$$u(t) = \Gamma(t) \text{ (Step of amplitude 1)}$$

$$\textcircled{1} \quad y_1(t) = A_1 e^{-t} + A_2 e^{-2t}$$

$$\textcircled{2} \quad y_2(t) = \frac{1}{2}$$

$$\textcircled{3} \quad y(t) = \frac{1}{2} + A_1 e^{-t} + A_2 e^{-2t}, \quad y'(t) = -A_1 e^{-t} - 2A_2 e^{-2t}$$

$$y(0) = 1 = \frac{1}{2} + A_1 + A_2, \quad y'(0) = -1 = -A_1 - 2A_2 \Rightarrow A_2 = \frac{1}{2}, \quad A_1 = 0$$

$$y(t) = \frac{1}{2} + \frac{1}{2} e^{-2t}$$

## 2. From Transfer function

- The transfer function does not take into account initial conditions.
- A way to consider them is to return to differential equation and take its Laplace transform with initial conditions. Summarizing

- 1 Obtain the differential equation from the transfer function. We have

$$G(s) = \frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} \Rightarrow \mathcal{L}^{-1} [D(s)Y(s)] = \mathcal{L}^{-1} [N(s)U(s)]$$

- 2 Take the Laplace transform of differential equation including initial conditions
- 3 Decompose  $Y(s)$  to obtain  $y(t)$

## 2. From Transfer function

### EXAMPLE (continued)

$$G(s) = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2} = \frac{Y(s)}{U(s)}$$
$$\Rightarrow y' + 2y = u$$

Then

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s} \Rightarrow$$
$$(s+2)Y(s) = \frac{1}{s} + 1 \Rightarrow$$
$$Y(s) = \frac{s+1}{s(s+2)} = \frac{1/2}{s} + \frac{1/2}{s+2} \Rightarrow$$

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}$$

### 3. From State equation

In that case the system is represented by

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad x(t_0)\end{aligned}$$

Introduce the transition matrix  $\Phi(t_0, t)$  defined by

$$x(t) = \Phi(t, t_0)x(t_0)$$

This matrix is unique and satisfies

- 1  $\Phi(t_0, t_0) = I$
- 2  $\frac{d\Phi(t, t_0)}{dt} = A\Phi(t, t_0)$
- 3  $\Phi(t_0, t_2) = \Phi(t_0, t_1)\Phi(t_1, t_2)$  (Semi-group property)
- 4  $\Phi(t_0, t)\Phi(t, t_0) = \Phi(t_0, t_0) = I \Rightarrow \Phi(t_0, t)^{-1} = \Phi(t, t_0)$

### 3. From State equation

Introduce the matrix function

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^i t^i}{i!} + \dots$$

$e^{A(t-t_0)}$  satisfies all the properties (Show it)

$$x(t) = \Phi(t_0, t)x(t_0) = e^{A(t-t_0)}x(t_0)$$

The response is given by (Show it using  $\frac{de^{A(t-t_0)}}{dt} = Ae^{A(t-t_0)} = e^{A(t-t_0)}A$  and  $e^{A(t-\tau)} = e^{A(t-t_0)}e^{A(t_0-\tau)}$ )

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

### 3. From State equation

Taking the Laplace transform of the state-space model, we have

$$\begin{aligned}sX(s) - x_0 &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

The response is given by

$$\begin{aligned}X(s) &= (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

and we can conclude that

$$e^{At} = \mathcal{L}^{-1} [(sI - A)^{-1}]$$

### 3. From State equation

#### PROPERTIES OF $e^{At}$

- $e^{At} = Me^{Jt}M^{-1}$  where  $J = M^{-1}AM$

- $$e^{\begin{pmatrix} \lambda_1 t & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n t \end{pmatrix}} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{pmatrix}$$

- $e^{At} = \mathcal{L}^{-1} [(sI - A)^{-1}]$

- $e^{A(t_2+t_1)} = e^{At_2}e^{At_1}$

- $e^{A_1+A_2} = e^{A_1}e^{A_2} = e^{A_2}e^{A_1}$  if  $A_1A_2 = A_2A_1$  because if  $A_1$  and  $A_2$  commute

$$(A_1 + A_2)^k = A_1^k + \binom{k}{1}A_1^{k-1}A_2 + \cdots + \binom{k}{k-1}A_1A_2^{k-1} + A_2^k$$

### 3. From State equation

#### EXAMPLE (continued)

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(t), \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{aligned} e^{At} &= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{pmatrix} \end{aligned}$$

### 3. From State equation

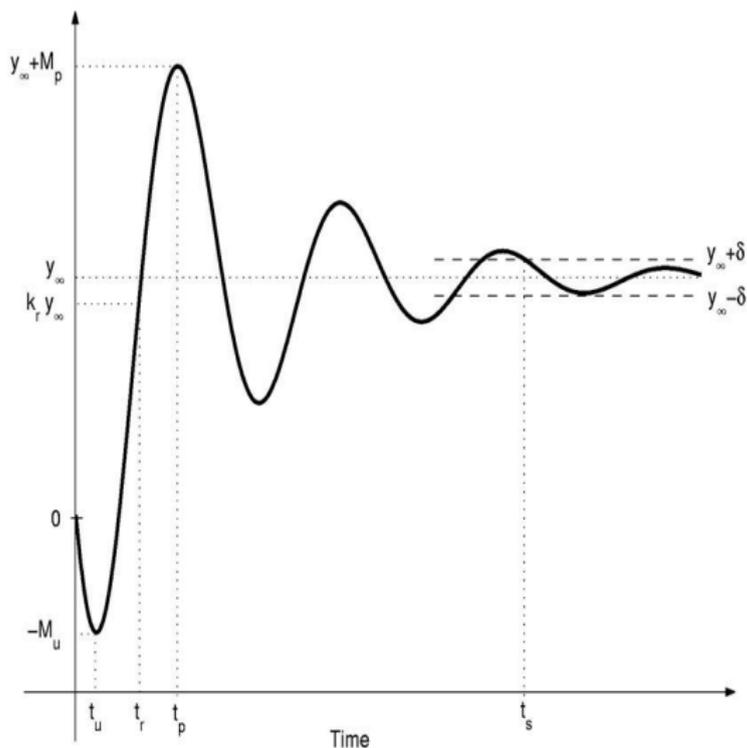
$$x(t) = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ + \int_0^t \begin{pmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 2e^{-2(t-\tau)} - 2e^{-(t-\tau)} & 2e^{-2(t-\tau)} - e^{-(t-\tau)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

$$x(t) = \begin{pmatrix} 3e^{-t} - e^{-2t} \\ 2e^{-2t} - 3e^{-t} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{\tau-t} - e^{2\tau-2t} \\ 2e^{2\tau-2t} - e^{\tau-t} \end{pmatrix} d\tau$$

$$y(t) = e^{-2t} + \int_0^t e^{2\tau-2t} d\tau = e^{-2t} + \left[ \frac{e^{2\tau-2t}}{2} \right]_0^t = \frac{1}{2} + \frac{1}{2}e^{-2t}$$

## 4. Step response

For a given system, in general the step response has the following form



## 4. Step response

We define the following indicators:

- 1 Steady state value,  $y_{\infty}$ : the final value of the step response (this is meaningless if the system has poles with  $\text{Re}[p_i] > 0$ ).
- 2 Rise time,  $t_r$ : The time elapsed up to the instant at which the step response reaches, for the first time, the value  $k_r y_{\infty}$ . The constant  $k_r$  varies from author to author, being usually either 0.9 or 1.
- 3 Overshoot,  $M_p$ : The maximum instantaneous amount by which the step response exceeds its final value. It is usually expressed as a percentage of  $y_{\infty}$ .
- 4 Undershoot,  $M_u$ : the (absolute value of the) maximum instantaneous amount by which the step response falls below zero.
- 5 Settling time,  $t_s$ : the time elapsed until the step response enters (without leaving it afterwards) a specified deviation band,  $\pm\delta$ , around the final value. This deviation  $\delta$ , is usually defined as a percentage of  $y_{\infty}$ , say 2% to 5%.

## 5. Frequency response

- Consider a linear system described by its differential equation or its transfer function  $G(s)$  whose poles are :  $p_1, p_2, \dots, p_n$  (distinct for simplicity).
- Suppose that the input  $u(t) = u_0 e^{j\omega_0 t}$ . The output is expressed by

$$y(t) = \sum_{i=1}^n A_i e^{p_i t} + |G(j\omega_0)| u_0 e^{j(\omega_0 t + \arg[G(j\omega_0)])}$$

Show that!

- Now if  $u(t) = u_0 \cos \omega_0 t$ , show that

$$y(t) = \sum_{i=1}^n A_i e^{p_i t} + |G(j\omega_0)| u_0 \cos(\omega_0 t + \arg [G(j\omega_0)])$$

If  $\text{Re}[p_i] < 0$  for all  $i = 1, \dots, n$ , the steady state response is

$$y(t) \rightarrow |G(j\omega_0)| u_0 \cos(\omega_0 t + \arg [G(j\omega_0)])$$

## 5. Frequency response - Bode diagram

- Bode diagrams consist of a pair of plots. One of these plots depicts the magnitude of the frequency response as a function of the angular frequency, and the other depicts the angle of the frequency response, also as a function of the angular frequency.
- Usually, Bode diagrams are drawn with special axes:
  - The abscissa axis is linear in  $\log(\omega)$  where the log is base 10. This allows a compact representation of the frequency response along a wide range of frequencies. The unit on this axis is the decade, where a decade is the distance between  $\omega_1$  and  $10\omega_1$  for any value of  $\omega_1$ .
  - The magnitude of the frequency response is measured in decibels [dB], i.e. in units of  $20 \log |G(j\omega)|$ . This has several advantages, including good accuracy for small and large values of  $|G(j\omega)|$ , facility to build simple approximations for  $20 \log |G(j\omega)|$ , and the fact that the frequency response of cascade systems can be obtained by adding the individual frequency responses.
  - The angle is measured on a linear scale in radians or degrees.
- Plot Bode diagrams for elementary first order transfer functions

## 5. Frequency response - Black (Nichols)

Another representation of frequency response consists of representing the modulus as a function of the angle. This the Black (or Nichols) diagram.

- The abscissa axis is linear and the angle is measured on a linear scale in radians or degrees.
- The magnitude of the frequency response is measured in decibels [dB], i.e. in units of  $20 \log G(j\omega)$ .
- For each value of the angle and magnitude corresponds a value of angular frequency on the obtained diagram. The diagram is oriented from low to high frequencies.

## 5. Frequency response - Nyquist diagram

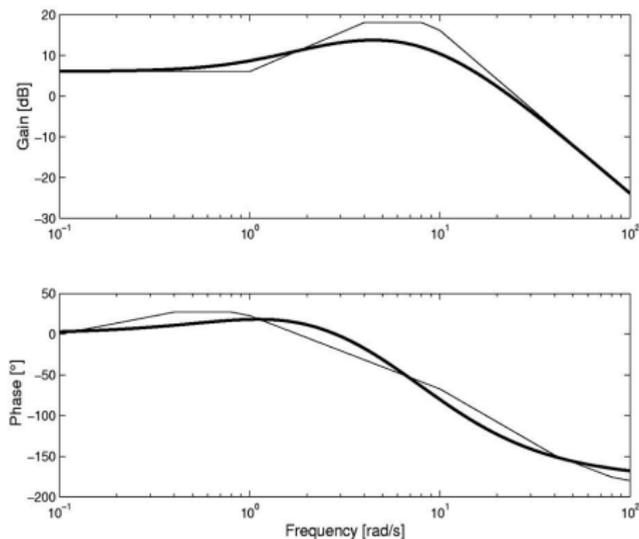
The last representation of frequency response consists of representing the real and imaginary parts of  $G(j\omega)$  as a function of the angular frequency in the complex plane. This is the Nyquist diagram.

- The X-axis is linear
- The Y-axis is linear
- For each value of the real and imaginary parts corresponds a value of angular frequency on the obtained diagram. The diagram is oriented from low to high frequencies.

## 5. Frequency response

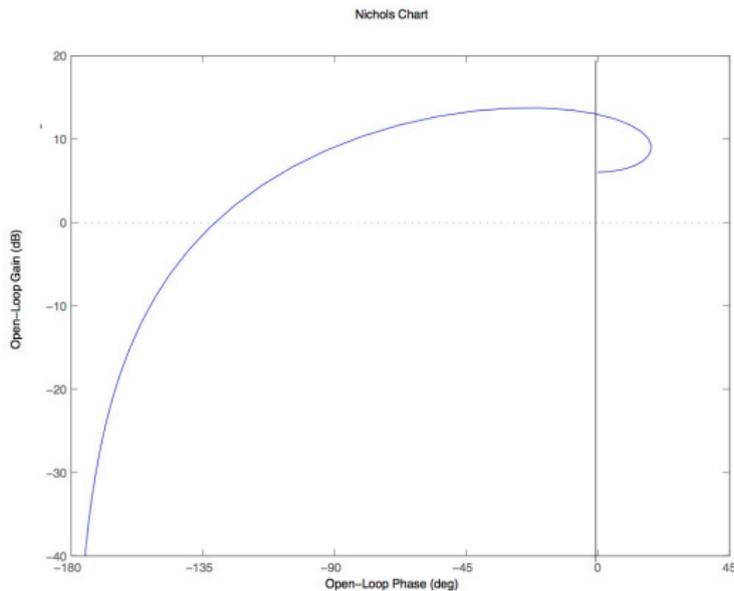
### EXAMPLE

$$G(s) = 640 \frac{s + 1}{(s + 4)(s + 8)(s + 10)}$$



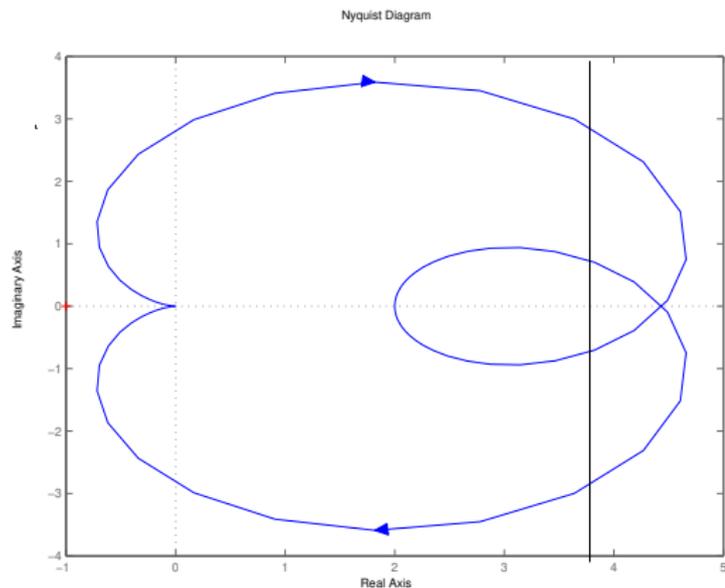
Bode plot

## 5. Frequency response



Nichols Chart

## 5. Frequency response



## 5. Frequency response

In general for frequency responses, we can define the following indicators

- 1 The static gain is the gain (when it is defined) corresponding to  $|G(j0)|$ ,
- 2 The bandwidth is defined as the frequency range in which the magnitude of  $G(j\omega)$  is equal to or larger than  $\frac{|G(j0)|}{\sqrt{2}}$ . The frequency  $\omega_c$  with the property  $|G(j\omega_c)| = \frac{|G(j0)|}{\sqrt{2}}$  is called the cutoff frequency. It corresponds to an attenuation of 3dB of the gain  $|G(j0)|$ .
- 3 The peak of resonance  $M_T$  is defined as the maximum of  $|G(j\omega)|$  attained at frequency  $\omega_T$  called the resonance frequency.

## 6. First order system

- In a canonic form, its transfer function is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{1 + Ts}, \quad K > 0 \text{ and } T > 0$$

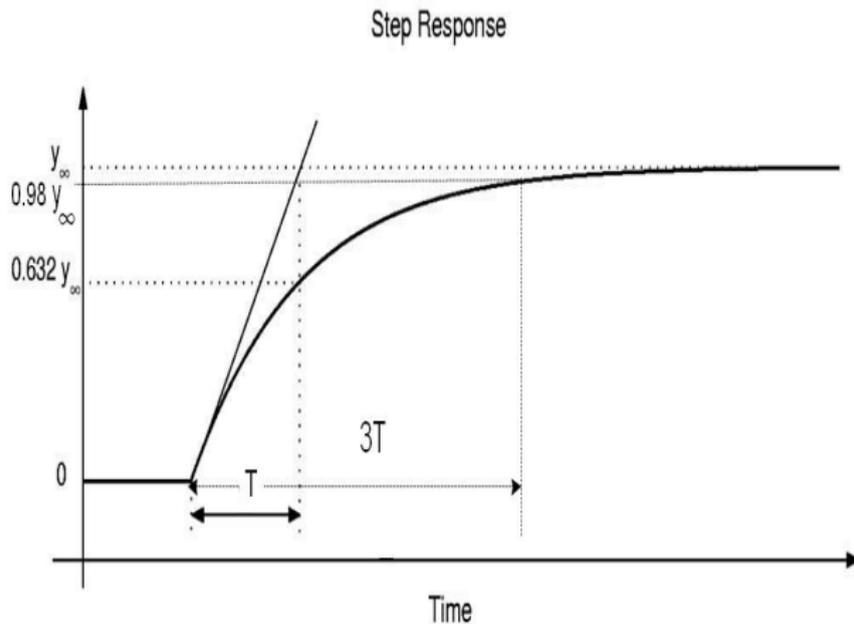
K is the static gain and T is the time constant

### Step response

For null initial conditions, the step response writes

$$y(t) = K\left(1 - e^{-\frac{t}{T}}\right)$$

## 6. First order system

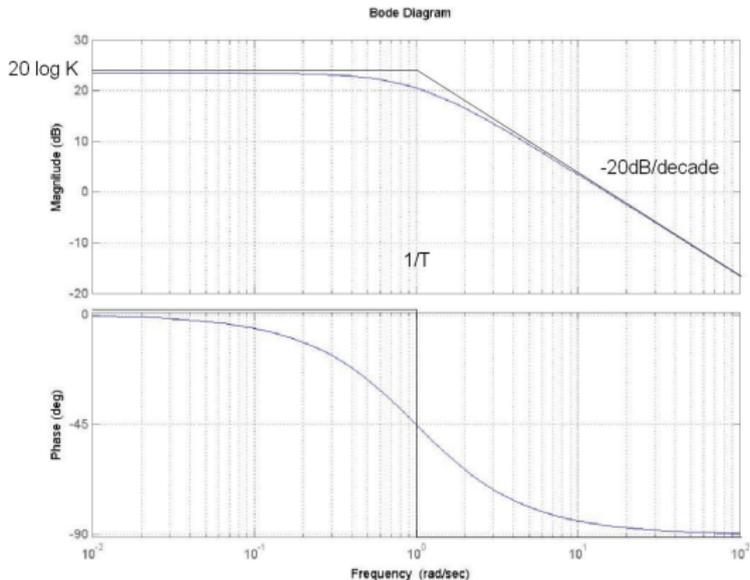


# 6. First order system

## Frequency response

The frequency response corresponds to

$$|G(j\omega)| = \frac{K}{\sqrt{1 + T^2\omega^2}}, \quad \varphi(\omega) = -\arctan T\omega$$



## 7. Second order system

In a canonical form, its transfer function is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2z\omega_n s + \omega_n^2},$$

$$K > 0, z > 0 \text{ and } \omega_n > 0$$

with

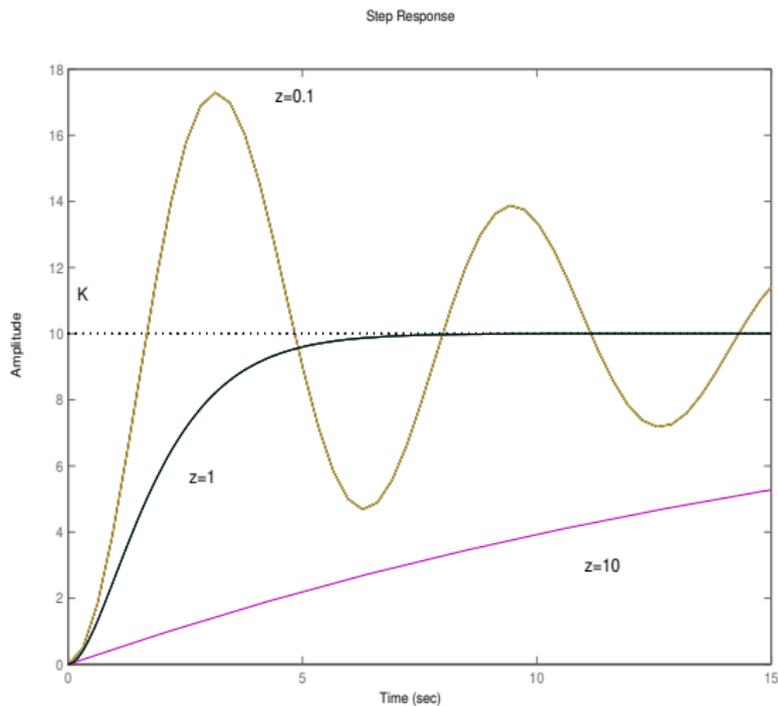
- $K$  : static gain
- $z$  : damping factor
- $\omega_n$  : natural frequency
- $\omega_d = \omega_n \sqrt{1 - z^2}$  : damped frequency if  $z < 1$

## 7. Second order system

### Step response

Damping ratio	Poles	Response $y(t)$
$z = 0$	$p_{1,2} = \pm j\omega_n$	$K(1 - \sin(\omega_n t + \frac{\pi}{2}))$
$0 < z < 1$	$p_{1,2} = -z\omega_n \pm j\omega_d$	$K(1 - \frac{\omega_n}{\omega_d} e^{-z\omega_n t} \sin(\omega_d t + \arccos z))$
$z = 1$	$p_{1,2} = -\omega_n$	$K - K(\omega_n t + 1)e^{-\omega_n t}$
$z > 1$	$p_{1,2} = -z\omega_n \pm \omega_n \sqrt{z^2 - 1}$	$K + \frac{Kp_2}{p_1 - p_2} e^{p_1 t} + \frac{Kp_1}{p_2 - p_1} e^{p_2 t}$

## 7. Second order system



## 7. Second order system

### Frequency response

We have

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2z\omega_n s + \omega_n^2}, \quad K > 0, z > 0 \text{ and } \omega_n > 0$$

$$|G(j\omega)| = \frac{K}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4z^2 \frac{\omega^2}{\omega_n^2}}}, \quad \varphi(\omega) = -\arctan \frac{\frac{2z\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

Damping ratio	Resonance	Static gain
$z = 0$	$\omega_r = \omega_n, M_r \text{ infinite}$	$K$
$0 < z < \frac{\sqrt{2}}{2}$	$\omega_r = \omega_n \sqrt{1 - 2z^2}, M_r = \frac{K}{2z\sqrt{1 - z^2}}$	$K$
$z > \frac{\sqrt{2}}{2}$	No resonance	$K$

## 7. Second order system

