CONTROL SYSTEMS DESIGN

Chapter II

Models for LTI Systems

Objective of Chapter II

- Introduce the main models for LTI systems (Differential equation, Transfert function and State-space model)
- Present the notion of state and the state-space models
- Discuss the relations between the models
- Introduce some canonical models

1. Main Assumptions

- The model has to capture the properties of the physical system
- Assumptions adopted in the sequel

1. **Linearity** (superposition principle)

$$\text{For all } \alpha_1 \text{ and } \alpha_2 \in \mathbb{R}, \; f(\alpha_1 u_1 + \alpha_1 u_2) = \alpha_1 f(u_1) + \alpha_2 f(u_2)$$

2. **Invariance** : the characteristics of the system are independent of the origin of time i.e. If $y(t) = f(u(t))$ then $\forall t_0$, $y(t - t_0) = f(u(t - t_0))$
3. **Causality**

$$\text{At time } t\text{, the output } y(t) \text{ only depends of inputs } u(t) \text{ at times } t_1 \leqslant t$$

4. We consider systems with **one input and one output** (Single Input, Single Output systems, **SISO**)

2. Differential equation

A linear time-invariant and causal system can be described by a linear differential equation

$$a_n \frac{d^{(n)}y(t)}{dt^n} + a_{n-1}\frac{d^{(n-1)}y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_m \frac{d^{(m)}u(t)}{dt^m} + b_{m-1}\frac{d^{(m-1)}u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

$$\text{Initial conditions} \quad y(0),\ y'(0), \cdots, y^{(n-1)}(0)$$

$m \leqslant n$ (causality), $a_i,\ b_i \in \mathbb{R}$ (Invariance)

n is the order of the system

2. Differential equation

Introducing the derivation operator:

$$r = \frac{d}{dt} \text{ and } r^i = \frac{d^i}{dt^i}$$

we have:

$$D(r)y(t) = N(r)u(t)$$

with:

$$D(r) = a_0 + a_1 r + \cdots + a_n r^n$$
$$N(r) = b_0 + b_1 r + \cdots + b_m r^m$$

$$\boxed{D(r) = 0, \text{ characteristic equation}}$$

The roots of $D(r) = 0$ are called the characteristic roots.

Note that because $D(r)$ is a polynomial with real coefficients The set of the characteristic roots is auto-conjugate meaning that it is invariant by conjugation or in other words, if λ is a complex characteristic root the $\bar{\lambda}$ is also a characteristic root. (Show it)

3. Transfer function

Introduce the unilateral Laplace Transform

$$\mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt = Y(s)$$

$$\mathcal{L}^{-1}[y(t)] = y(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} Y(s)e^{st}ds$$

Define

$$\sigma \triangleq \inf\{\alpha \in \mathbb{R} : |y(t)|e^{-\alpha t} \text{is integrable}\}$$

The Laplace transform is defined for $Re[s] > \sigma$.

3. Transfer function

PROPERTIES

$f(t)$	$\mathcal{L}[f(t)]$	Names
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$	Linear combination
$\frac{dy(t)}{dt}$	$sY(s) - y(0^-)$	Derivative Law
$\frac{d^k y(t)}{dt^k}$	$s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left. \frac{d^{i-1} y(t)}{dt^{i-1}} \right\|_{t=0^-}$	High order derivative
$\int_{0^-}^{t} y(\tau) d\tau$	$\frac{1}{s} Y(s)$	Integral Law
$y(t-\tau)\mu(t-\tau)$	$e^{-s\tau} Y(s)$	Delay
$t y(t)$	$-\frac{dY(s)}{ds}$	
$t^k y(t)$	$(-1)^k \frac{d^k Y(s)}{ds^k}$	
$\int_{0^-}^{t} f_1(\tau) f_2(t-\tau) d\tau$	$F_1(s) F_2(s)$	Convolution
$\lim_{t \to \infty} y(t)$	$\lim_{s \to 0} sY(s)$	Final Value Theorem
$\lim_{t \to 0^+} y(t)$	$\lim_{s \to \infty} sY(s)$	Initial Value Theorem
$f_1(t) f_2(t)$	$\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F_1(\zeta) F_2(s-\zeta) d\zeta$	Time domain product
$e^{at} f_1(t)$	$F_1(s-a)$	Frequency Shift

TABLE OF COMMON LAPLACE TRANSFORMS

$f(t) \quad (t \geq 0)$	$\mathcal{L}[f(t)]$	Region of Convergence
1	$\frac{1}{s}$	$\sigma > 0$
$\delta_D(t)$	1	$\|\sigma\| < \infty$
t	$\frac{1}{s^2}$	$\sigma > 0$
$t^n \quad n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$e^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{s-\alpha}$	$\sigma > \Re\{\alpha\}$
$te^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{(s-\alpha)^2}$	$\sigma > \Re\{\alpha\}$
$\cos(\omega_o t)$	$\frac{s}{s^2+\omega_o^2}$	$\sigma > 0$
$\sin(\omega_o t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\sigma > 0$
$e^{\alpha t}\sin(\omega_o t+\beta)$	$\frac{(\sin\beta)s+\omega_o^2\cos\beta-\alpha\sin\beta}{(s-\alpha)^2+\omega_o^2}$	$\sigma > \Re\{\alpha\}$
$t\sin(\omega_o t)$	$\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$t\cos(\omega_o t)$	$\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$\mu(t)-\mu(t-\tau)$	$\frac{1-e^{-s\tau}}{s}$	$\|\sigma\| < \infty$

3. Transfer function

Taking the Laplace transform of the differential equation leads to

$$\underbrace{(a_0 + a_1 s + \cdots + a_n s^n)}_{D(s)} Y(s) = \underbrace{(b_0 + b_1 s + \cdots + b_m s^m)}_{N(s)} U(s) + I(s)$$

with $I(s) = 0$ for null initial conditions

Then

$$\boxed{Y(s) = \underbrace{\frac{N(s)}{D(s)}}_{G(s)} U(s) + \frac{I(s)}{D(s)}}$$

- $G(s)$ is called the transfer function if $N(s)$ and $D(s)$ **are coprime polynomials**
- $D(s) = 0$ is the **characteristic polynomial**.
- The roots of $N(s)$ are the **zeros of the system**
- The roots of $D(s)$ are the **poles of the system**.

4. State Space equations

- It is well known that the input/output relations are not sufficient to determine the behavior of a system. For the differential equation, we need the initial conditions.
- A dynamical system is then characterized by a set of n variables called the **state** and denoted by a vector $x(t)$ which belongs to a vector space (Linear space) $\mathcal{X}$ called **the state space** and the idea consists in parametrizing the input/output relations with respect to the state.
- Another way more axiomatic consists in defining the notions of state and state transition function and deduce the required properties to characterize a dynamical system (Group properties)
- For LTI systems, a state-space model can be described by the following equations:

$$\boxed{\begin{array}{rcl} \dfrac{dx(t)}{dt} & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t), \quad x(0) = x_0 \end{array}}$$

where A, B, C, D are matrices such that
$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ $(m = 1)$, $C \in \mathbb{R}^{p\times n}$ $(p = 1)$ and $D \in \mathbb{R}^{m\times p} (\in \mathbb{R})$

4. State Space equations

RELATIONS WITH TRANSFER FUNCTION

Taking the Laplace transform, we obtain:

$$\begin{aligned} sX(s) - x_0 &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

and hence:

$$X(s) = (sI - A)^{-1}BU(s)(s) + (sI - A)^{-1}x_0$$

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) + C(sI - A)^{-1}x_0 = G(s)U(s) + \frac{I(s)}{D(s)}$$

$$\boxed{\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ \frac{I(s)}{D(s)} &= C(sI - A)^{-1}x_0 \end{aligned}}$$

4. State Space equations

We have:

$$(sI - A)^{-1} = \frac{\mathrm{Cof}((sI - A))^T}{\det(sI - A)}$$

where $\mathrm{Cof}((sI - A))$ is the cofactor matrix associated to $sI - A$. Then

$$G(s) = \frac{C\ \mathrm{Cof}((sI - A))^T B + D \det(sI - A)}{\det(sI - A)}$$

- If $D = 0$, $\deg(N(s)) < \deg(D(s))$.
- If $D \neq 0$, $\deg(N(s)) = \deg(D(s))$.
- The characteristic polynomial is given by $\det(sI - A) = 0$
- The poles are the **eigenvalues of matrix** A.
- The zeros are the roots of $C\ \mathrm{Cof}((sI - A))^T B + D \det(sI - A) = 0$

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4. State Space equations

NO UNICITY

We can change the state vector by:

$$x = Mz \text{ with } M \text{ invertible}$$

$$\frac{dx(t)}{dt} = M\frac{dz(t)}{dt} = AMz(t) + Bu(t)$$

The new state-space model writes

$$\begin{array}{rcl} \dfrac{dz(t)}{dt} & = & M^{-1}AMz(t) + M^{-1}Bu(t) \\ y(t) & = & CMz(t) + Du(t) \end{array}$$

The transfer function can also be deduced from $((S_1S_2)^{-1} = S_2^{-1}S_1^{-1})$:

$$\begin{aligned} G(s) &= CM(sI - M^{-1}AM)^{-1}M^{-1}B + D \\ &= C(sMM^{-1} - MM^{-1}AMM^{-1})^{-1}B + D \\ &= C(sI - A)^{-1}B + D \text{ for all } M \text{ invertible} \end{aligned}$$

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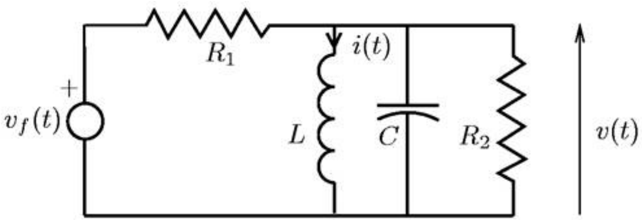
5. An Example



$u(t) = v_f(t)$ and $y(t) = v(t)$. We can write:

$$y(t) = L\frac{di(t)}{dt}$$

$$\frac{u(t) - y(t)}{R_1} = i(t) + C\frac{dy(t)}{dt} + \frac{y(t)}{R_2}$$

DIFFERENTIAL EQUATION

Differentiating the second equation, we obtain:

$$\frac{1}{R_1}(\frac{du(t)}{dt} - \frac{dy(t)}{dt}) = \frac{di(t)}{dt} + C\frac{d^2y(t)}{dt^2} + \frac{1}{R_2}\frac{dy(t)}{dt}$$

The differential equation can be deduced:

$$\boxed{\frac{d^2y(t)}{dt^2} + \left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = +\frac{1}{R_1 C}\frac{du(t)}{dt}}$$

The order is 2 and the characteristic equation is:

$$r^2 + \left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)r + \frac{1}{LC} = 0$$

5. An Example

TRANSFER FUNCTION

Taking the Laplace transform of the differential equation or directly from elementary equation, we obtain:

$$\boxed{G(s) = \frac{\dfrac{s}{R_1 C}}{s^2 + \left(\dfrac{1}{R_1 C} + \dfrac{1}{R_2 C}\right) s + \dfrac{1}{LC}}}$$

The characteristic polynomial is given by:

$$s^2 + \left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right) s + \frac{1}{LC} = 0$$

STATE SPACE

These equations can be rearranged as:

$$\frac{di(t)}{dt} = \frac{y(t)}{L}$$
$$\frac{dy(t)}{dt} = -\frac{i(t)}{C} - \left[\frac{1}{R_1 C} + \frac{1}{R_2 C}\right] y(t) + \frac{u(t)}{R_1 C}$$

Letting $x(t) = \begin{pmatrix} i(t) & y(t) \end{pmatrix}'$, we obtain:

$$\begin{aligned} \frac{dx(t)}{dt} &= \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\left[\frac{1}{R_1 C} + \frac{1}{R_2 C}\right] \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{R_1 C} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

5. An Example

The transfer function can be deduced from the state space equation.

$$G(s) = C(sI - A)^{-1}B =$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s + \left[\frac{1}{R_1C} + \frac{1}{R_2C}\right] \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{R_1C} \end{bmatrix}$$

$$= \frac{1}{s^2 + \left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)s + \frac{1}{LC}} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + \left[\frac{1}{R_1C} + \frac{1}{R_2C}\right] & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{R_1C} \end{bmatrix}$$

$$\boxed{G(s) = \frac{\frac{s}{R_1C}}{s^2 + \left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)s + \frac{1}{LC}}}$$

5. An Example

Now consider that $R_1 = 1, \; , R_2 = 2, \; , C = 0.5, \; L = 1$. We have :

$$\begin{aligned} \frac{dx(t)}{dt} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

Consider the new state vector :

$$x = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} z \;\Rightarrow\; z = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} x$$

$$M^{-1}AM = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

5. An Example

$$M^{-1}B = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$CM = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{dz(t)}{dt} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -1 & -2 \end{bmatrix} z(t) \end{aligned}}$$

6. Linearization

In fact, linearity never occurs in practice. A state space description is given by:

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

In some situations, (regulation), we can used a linearized model. Suppose that $x_e(t)$, $u_e(t)$ and $y_e(t)$ satisfy:

$$\begin{aligned} \frac{dx_e(t)}{dt} &= f(x_e(t), u_e(t)) \\ y_e(t) &= g(x_e(t), u_e(t)) \end{aligned}$$

Then, we have:

$$\frac{dx(t)}{dt} \approx f((x_e(t), u_e(t)) + \left.\frac{\partial f}{\partial x}\right|_{(x_e, u_e)} (x(t) - x_e(t)) + \left.\frac{\partial f}{\partial u}\right|_{(x_e, u_e)} (u(t) - u_e(t))$$

$$y(t) \approx g(x_e(t), u_e(t)) + \left.\frac{\partial g}{\partial x}\right|_{(x_e, u_e)} (x(t) - x_e(t)) + \left.\frac{\partial g}{\partial u}\right|_{(x_e, u_e)} (u(t) - u_e(t))$$

6. Linearization

Denoting by:

$$\Delta x(t) = x(t) - x_e(t), \quad \Delta y(t) = y(t) - y_e(t) \text{ and } \Delta u(t) = u(t) - u_e(t)$$

we obtain:

$$\boxed{\begin{array}{rcl} \dfrac{d\Delta x(t)}{dt} & = & A(t)\ \Delta x(t) + B(t)\ \Delta u(t) \\ \Delta y(t) & = & C(t)\ \Delta x(t) + D(t)\ \Delta u(t) \end{array}}$$

with

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{(x_e(t),u_e(t))}, \quad B(t) = \left.\frac{\partial f}{\partial u}\right|_{(x_e(t),u_e(t))}$$

$$C(t) = \left.\frac{\partial g}{\partial x}\right|_{(x_e(t),u_e(t))}, \quad D(t) = \left.\frac{\partial g}{\partial u}\right|_{(x_e(t),u_e(t))}$$

If $x_e(t)$, $u_e(t)$ and $y_e(t)$ are constant, then $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are also constant.

6. Linearization

EXAMPLE

Consider the following system

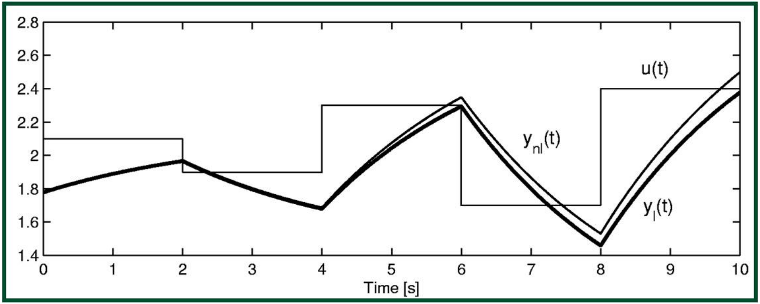
$$\frac{dx(t)}{dt} = -\sqrt{x(t)} + \frac{u^2(t)}{3} = f(x(t), u(t))$$
$$y(t) = x(t)$$

u moves around 2. Then $u_e = 2$, and $\frac{dx(t)}{dt} = 0$ lead to $\sqrt{x_e(t)} = \frac{u_e^2}{3} = \frac{4}{3}$. Then

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e(t), u_e(t))} = -\left.\frac{1}{2\sqrt{x(t)}}\right|_{(\frac{4}{3}, 2)} = -\frac{3}{8}$$
$$B = \left.\frac{\partial f}{\partial u}\right|_{(x_e(t), u_e(t))} = \frac{2u_e}{3} = \frac{4}{3}$$

Then

$$\frac{d\Delta x(t)}{dt} = A(t)\ \Delta x(t) + B(t)\ \Delta u(t) = -\frac{3}{8}\Delta x(t) + \frac{4}{3}\Delta u(t)$$
$$\Delta y(t) = \Delta x(t)$$



7. Transfer function $\mapsto$ State equation

A in diagonal form

We know $G(s)$ and we want to obtain a state equation. We suppose that the poles are different (single), real and deg $N(s) < n$.

$$G(s) = \frac{N(s)}{\prod_{i=1}^{n}(s-p_i)} = \sum_{i=1}^{n}\frac{\alpha_i}{(s-p_i)} \Rightarrow Y(s) = \sum_{i=1}^{n}\underbrace{\frac{\alpha_i U(s)}{(s-p_i)}}_{X_i}$$

We obtain

$$\begin{array}{rcl} \dfrac{dx}{dt} & = & \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_n \end{bmatrix} x + \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} u \\ y & = & \begin{bmatrix} 1 & \cdots & \cdots & 1 \end{bmatrix} x \end{array}$$

7. Transfer function $\mapsto$ State equation

Suppose now that a pole is multiple of order r.

$$G(s) = \frac{N(s)}{(s-p)^r} \Rightarrow$$

$$Y(s) = \alpha_1 \underbrace{\frac{U(s)}{(s-p)}}_{X_r} + \alpha_2 \underbrace{\frac{U(s)}{(s-p)^2}}_{X_{r-1}} + \cdots + \alpha_r \underbrace{\frac{U(s)}{(s-p)^r}}_{X_1}$$

$$\boxed{\begin{array}{rcl} \dfrac{dx}{dt} & = & \begin{bmatrix} p & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & p & 1 \\ 0 & \cdots & 0 & p \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y & = & \begin{bmatrix} \alpha_r & \cdots & \alpha_2 & \alpha_1 \end{bmatrix} x \end{array}}$$

A is a Jordan Block of order r.

7. Transfer function $\mapsto$ State equation

Complex poles

Suppose that we have the following transfer function

$$G(s) = \frac{as+b}{(s+\alpha)^2+\beta^2} = \frac{as+b}{(s+\alpha-i\beta)(s+\alpha+i\beta)} = \frac{\dfrac{a\beta+i(b-a\alpha)}{2\beta}}{s+\alpha+i\beta} + \frac{\dfrac{a\beta-i(b-a\alpha)}{2\beta}}{s+\alpha-i\beta}$$

A state space equation can be obtained as

$$\boxed{\begin{array}{rcl} \dfrac{dx}{dt} & = & \begin{bmatrix} -\alpha-i\beta & 0 \\ 0 & -\alpha+i\beta \end{bmatrix} x + \begin{bmatrix} \dfrac{a\beta+i(b-a\alpha)}{2\beta} \\ \dfrac{a\beta-i(b-a\alpha)}{2\beta} \end{bmatrix} u \\ y & = & \begin{bmatrix} 1 & 1 \end{bmatrix} x \end{array}}$$

7. Transfer function $\mapsto$ State equation

Let consider a new state vector

$$x = \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} z \Rightarrow z = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} x$$

$$M^{-1}AM = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\alpha - i\beta & 0 \\ 0 & i\beta - \alpha \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{bmatrix}$$

$$M^{-1}B = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{a\beta + i(b - a\alpha)}{2\beta} \\ \dfrac{a\beta - i(b - a\alpha)}{2\beta} \end{bmatrix} = \begin{bmatrix} \dfrac{b - a\alpha}{\beta} \\ a \end{bmatrix}$$

7. Transfer function $\mapsto$ State equation

$$CM = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{dz}{dt} &= \begin{bmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{bmatrix} z + \begin{bmatrix} \dfrac{b - a\alpha}{\beta} \\ a \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} z \end{aligned}}$$

7. Transfer function $\mapsto$ State equation

EXAMPLES

$$G(s) = \frac{1}{s(s-1)(s+1)} = -\frac{1}{s} + \frac{1/2}{(s-1)} + \frac{1/2}{(s+1)}$$

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x \end{aligned}$$

7. Transfer function $\mapsto$ State equation

$$G(s) = \frac{1}{s^3(s+1)^2(s-1)} = -\frac{2}{s} + \frac{1}{s^2} - \frac{1}{s^3} + \frac{7/4}{(s+1)} + \frac{1/2}{(s+1)^2} + \frac{1/4}{(s-1)}$$

$$\begin{aligned}
\frac{dx}{dt} &= \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & -1 & 1 & \\ & & & 0 & -1 & \\ & & & & & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1/4 \end{bmatrix} u \\
y &= \begin{bmatrix} -1 & 1 & -2 & 1/2 & 7/4 & 1 \end{bmatrix} x
\end{aligned}$$

7. Transfer function $\mapsto$ State equation

$$G(s) = \frac{s-1}{(s+1)^2(s^2+2s+2)} = \frac{1}{s+1} + \frac{-2}{(s+1)^2} + \frac{1-s}{(s+1)^2+1}$$

$$\alpha = 1 \quad \beta = 1 \quad a = -1$$
$$b = 1 \quad \frac{b - \alpha a}{\beta} = 2$$

$$\boxed{\begin{array}{rcl} \dfrac{dx}{dt} & = & \begin{bmatrix} -1 & 1 & & \\ 0 & -1 & & \\ & & -1 & -1 \\ & & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} u \\ y & = & \begin{bmatrix} -2 & 1 & 0 & 1 \end{bmatrix} x \end{array}}$$

A in companion form (controller and observer forms)

$$G(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}, \quad m < n$$

We can obtain

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{b_0}{a_n} & \frac{b_1}{a_n} & \cdots & 0 \end{bmatrix} x$$

This is "the controller form". Try to show that.

7. Transfer function $\mapsto$ State equation

$$
\begin{aligned}
\frac{d\bar{x}}{dt} &= \begin{bmatrix} -\frac{a_{n-1}}{a_n} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{a_1}{a_n} & \ddots & 0 & 1 \\ -\frac{a_0}{a_n} & 0 & \cdots & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ \vdots \\ \frac{b_1}{a_n} \\ \frac{b_0}{a_n} \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \bar{x}
\end{aligned}
$$

This is the "observer form". Try to show that (more involved).

7. Transfer function $\mapsto$ State equation

EXAMPLES

$$G(s)\frac{1}{s(s-1)(s+1)} = \frac{1}{s^3 - s}$$

$$a_0 = 0 \quad a_1 = -1 \quad a_2 = 1 \quad b_0 = 1$$

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{aligned}$$

$$\begin{aligned} \frac{dz}{dt} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} z \end{aligned}$$

7. Transfer function $\mapsto$ State equation

$$G(s) = \frac{1}{2s^3(s+1)^2(s-1)} = \frac{1}{2s^6 + 2s^5 - 2s^4 - 2s^3}$$

$$a_0 = 0 \quad a_1 = 0 \quad a_2 = 0 \quad a_3 = -2 \quad a_4 = -2 \quad a_5 = 2 \quad a_6 = 2 \quad b_0 = 1$$

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x \end{aligned}$$

7. Transfer function $\mapsto$ State equation

$$G(s) = \frac{1}{2s^3(s+1)^2(s-1)} = \frac{1}{2s^6 + 2s^5 - 2s^4 - 2s^3}$$

$$a_0 = 0 \quad a_1 = 0 \quad a_2 = 0 \quad a_3 = -2 \quad a_4 = -2 \quad a_5 = 2 \quad a_6 = 2 \quad b_0 = 1$$

$$\begin{aligned}
\frac{dz}{dt} &= \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} z
\end{aligned}$$

7. Transfer function $\mapsto$ State equation

$$G(s) = \frac{s-1}{(s+1)^2(s^2+2s+2)} = \frac{s-1}{s^4+4s^3+5s^2+6s+2}$$

$$a_0 = 2 \quad a_1 = 6 \quad a_2 = 5 \quad a_3 = 4 \quad a_4 = 1 \qquad b_0 = -1 \quad b_1 = 1$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -6 & -5 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} x \end{aligned}}$$

7. Transfer function $\mapsto$ State equation

$$G(s) = \frac{s-1}{(s+1)^2(s^2+2s+2)} = \frac{s-1}{s^4+4s^3+5s^2+6s+2}$$

$$a_0 = 2 \quad a_1 = 6 \quad a_2 = 5 \quad a_3 = 4 \quad a_4 = 1 \qquad b_0 = -1 \quad b_1 = 1$$

$$\begin{aligned} \frac{dz}{dt} &= \begin{bmatrix} -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} z \end{aligned}$$

7. Transfer function $\mapsto$ State equation

$$G(s) = \frac{s-1}{(s+1)^2(s^2+2s+2)} = \frac{s-1}{s^4+4s^3+5s^2+6s+2}$$

$$a_0 = 2 \quad a_1 = 6 \quad a_2 = 5 \quad a_3 = 4 \quad a_4 = 1 \qquad b_0 = -1 \quad b_1 = 1$$

$$\boxed{\begin{array}{rcl} \dfrac{dz}{dt} & = & \begin{bmatrix} -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u \\ y & = & \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} z \end{array}}$$

7. Transfer function $\mapsto$ State equation

Case $m = n$

For proper systems ($D \neq 0$), we have :

$$G(s) = \frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \underbrace{\frac{R(s)}{D(s)}}_{(A,B,C)} + \underbrace{Q}_{D}$$

Q: quotient, $R(s) = N(s) - QD(s)$: rest.

Example:

$$G(s) = \frac{2s^2 + 7s + 7}{s^2 + 3s + 2} = \frac{2(s^2 + 3s + 2)}{s^2 + 3s + 2} + \frac{s + 3}{s^2 + 3s + 2}$$

$$\begin{cases} \dfrac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 3 & 1 \end{bmatrix} x + 2u \end{cases}$$

8. Differential equation $\mapsto$ State equation

Case $m = 0$

Select $x_1 = y,\ x_2 = y', \cdots, x_n = y^{(n-1)}$, we obtain

$$
\begin{aligned}
\frac{dx}{dt} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{b_0}{a_n} \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x
\end{aligned}
$$

8. Differential equation $\mapsto$ State equation

Case $m > 0$

We select the following state variables

$$\begin{aligned} x_1 &= y - \alpha_n u \\ x_2 &= x_1' - \alpha_{n-1} u \\ x_3 &= x_2' - \alpha_{n-2} u \\ &\vdots \\ x_n &= x_{n-1}' - \alpha_1 u \end{aligned}$$

8. Differential equation $\mapsto$ State equation

we have

$$
\begin{aligned}
x_1' &= x_2 + \alpha_{n-1}u = y' - \alpha_n u' \\
x_2' &= x_3 + \alpha_{n-2}u = x_1'' - \alpha_{n-1}u' = y'' - \alpha_n u'' - \alpha_{n-1}u' \\
x_3' &= x_4 + \alpha_{n-3}u = x_2'' - \alpha_{n-2}u' = y''' - \alpha_n u''' - \alpha_{n-1}u'' - \alpha_{n-2}u' \\
&\vdots \\
x_{n-1}' &= x_n + \alpha_1 u = x_{n-2}'' - \alpha_2 u' \\
&= y^{(n-1)} - \alpha_n u^{(n-1)} - \alpha_{n-1}u^{(n-2)} - \alpha_{n-2}u^{(n-3)} - \cdots - \alpha_2 u' \\
x_n' &= y^{(n)} - \alpha_n u^{(n)} - \alpha_{n-1}u^{(n-1)} - \alpha_{n-2}u^{(n-2)} - \cdots - \alpha_2 u'' - \alpha_1 u' \\
&= -\frac{a_0}{a_n}y - \frac{a_1}{a_n}y' - \cdots - \frac{a_{n-1}}{a_n}y^{(n-1)} + \frac{b_0}{a_n}u - \cdots + \frac{b_m}{a_n}u^{(m)} \\
&\quad - \alpha_n u^{(n)} - \alpha_{n-1}u^{(n-1)} - \alpha_{n-2}u^{(n-2)} - \cdots - \alpha_2 u'' - \alpha_1 u'
\end{aligned}
$$

8. Differential equation $\mapsto$ State equation

Replacing the derivatives of y an selecting α_i, $i = 1, \cdots, n$, in order to eliminate the derivatives of u, this last equation writes

$$x_n' = -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_n + \alpha_0 u$$

The state equation becomes

$$\begin{array}{rcl} \dfrac{dx}{dt} & = & \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 1 \\ -\dfrac{a_0}{a_n} & -\dfrac{a_1}{a_n} & \cdots & -\dfrac{a_{n-1}}{a_n} \end{bmatrix} x + \begin{bmatrix} \alpha_{n-1} \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{bmatrix} u \\ y & = & \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x + \alpha_n u \end{array}$$

8. Differential equation $\mapsto$ State equation

EXAMPLE

$$y'' + 3y' + 2y = u' + u$$

We select the state variables

$$\begin{aligned} x_1 &= y - \alpha_2 u \\ x_2 &= x_1' - \alpha_1 u = y' - \alpha_2 u' - \alpha_1 u \end{aligned}$$

We have

$$\begin{aligned} x_1' &= x_2 + \alpha_1 u \\ x_2' &= x_1'' - \alpha_1 u' = y'' - \alpha_2 u'' - \alpha_1 u' \\ &= -3y' - 2y + u' + u - \alpha_2 u'' - \alpha_1 u' \end{aligned}$$

8. Differential equation $\mapsto$ State equation

But

$$y = x_1 + \alpha_2 u$$
$$y' = x_2 + \alpha_2 u' + \alpha_1 u$$

Replacing in the previous equation, leads to

$$x_2' = -2x_1 - 3x_2 + (1 - 3\alpha_1 - 2\alpha_2)u + (1 - \alpha_1 + 3\alpha_2)u' + \alpha_2 u''$$

Then we select

$$\alpha_2 = 0 \text{ and } \alpha_1 = 1 \text{ leading to } x_2' = -2x_1 - 3x_2 - 2u$$

The corresponding state space model is given by

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}}$$

8. State equation $\mapsto$ State equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t), \quad x(0) = x_0$$

$$x(t) = Mz(t) \Longrightarrow$$

$$\frac{dz(t)}{dt} = M^{-1}AMz(t) + M^{-1}Bu(t)$$
$$y(t) = CMz(t) + Du(t), \quad z(0) = M^{-1}x(0)$$

We can change the state equation for obtaining a state equation with a special form for the dynamical matrix

- Diagonal form
- Controller form
- Observer form
- ...

DIAGONAL FORM

The problem of diagonalizing a square matrix A of order n consists in determining vectors v_i and numbers λ_i, solutions of the following equation :

$$A\underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{M} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

8. State equation $\mapsto$ State equation

Vectors v_i and numbers λ_i must verify

$$\boxed{Av_i = \lambda_i v_i \text{ or similarly } (\lambda_i I - A)v_i = 0}$$

and are called respectively *eigenvector* and *eigenvalue*. A solution $v_i \neq 0$ exists if

$$\boxed{\det(\lambda_i I - A) = 0}$$

$P(\lambda) = \det(\lambda I - A)$ is a polynomial of order n, *the characteristic polynomial*.

- If all the roots are distinct, the matrix is diagonalizable and the eigenvectors are obtained solving $Av_i = \lambda_i v_i$ for $i = 1, \ldots, n$.
- If there exists a multiple eigenvalue of order r (r: *algebraic multiplicity*)
  - If the eigenspace $Av_i = \lambda_i v_i$ is of dimension r and the matrix is diagonalizable
  - If the eigenspace $Av_i = \lambda_i v_i$ is of dimension $l < r$ (l : *geometric multiplicity*). The matrix is not diagonizable but can be put in a canonical form called a **Jordan form**.
- **The Jordan canonical form is unique**

8. State equation $\mapsto$ State equation

- More precisely, this canonical form exhibits l Jordan blocks associated with λ_i.
- A Jordan block of order j associated to λ_i is a $j \times j$ matrix having the following generic form

$$\begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

- To the multiple eigenvalue of order r , are associated l Jordan blocks whose the sum of orders is equal to r
- For example, if λ_i is an eigenvalue of order $r = 4$ and if $l = 2$, the possible Jordan structures are

$$\begin{bmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 \\ 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \lambda_i \end{bmatrix} \quad \begin{bmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & \lambda_i \end{bmatrix}$$

8. State equation $\mapsto$ State equation

To compute the change of basis leading to the Jordan form (which is unique for a given matrix A), we introduce the notion of generalized eigenvector.

Definition

Vectors $v_1, \dots, v_r$ are called **a chain of eigenvectors of order** r associated with an eigenvalue λ of a matrix A if :

$$\begin{aligned} v_r &= v \\ v_{r-1} &= (A - \lambda I)v_r \\ v_{r-2} &= (A - \lambda I)v_{r-1} \\ &\dots \\ v_2 &= (A - \lambda I)v_3 \\ v_1 &= (A - \lambda I)v_2 \end{aligned}$$

where v is called a **generalized eigenvector** and satisfies: $(A - \lambda I)^r v = 0$.

8. State equation $\mapsto$ State equation

From the previous definition, we can write :

$$A\underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_{r-1} & v_r \end{bmatrix}}_{M} = \begin{bmatrix} \lambda v_1 & \lambda v_2 + v_1 & \cdots & \lambda v_{r-1} + v_{r-2} & \lambda v_r + v_{r-1} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_{r-1} & v_r \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}}_{J}$$

$$\Rightarrow M^{-1}AM = J$$

To obtain the Jordan form

1. Determine the eigenvalues of A
2. Find a basis for each eigenspace
3. If necessary complete the basis. If the multiplicity of the eigenvalue λ_i is equal to r_i while the dimension of the associated eigenspaces is equal to $l_i < r$ meaning that a basis is defined by l_i vectors $\underbrace{v_1,\ w_1, ..., z_1}_{l_i \text{ vectors}}$. $r_i - l_i$ vectors have to be determined for completing the basis and obtain r_i independent vectors. This can be done by generating the associated chain of eigenvectors (expressed for example v_1):
$$(A - \lambda_i I)v_2 = v_1,\ \ (A - \lambda_i I)v_3 = v_2, \dots$$
A new determined vector v_i must be independent of the previous ones.
4. A change of basis leading to the Jordan form is:
$$M = [v_1,\ v_2, \cdots, w_1,\ w_2, \cdots, z_1,\ z_2, \cdots]$$

EXAMPLES

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \lambda I - A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix} \rightarrow \det(\lambda I - A) = \lambda^3 - \lambda$$

$$(\lambda I - A)v = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\lambda - y \\ y\lambda - z \\ z\lambda - y \end{bmatrix}$$

$$\lambda = 0 \quad y = z = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 1 \qquad \begin{bmatrix} x-y \\ y-z \\ z-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = y = z \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1 \qquad \begin{bmatrix} -x-y \\ -y-z \\ -z-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = -y = z \Rightarrow v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

8. State equation $\mapsto$ State equation

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \lambda I - A = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & 2 \\ 0 & 0 & \lambda + 1 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda + 1)(\lambda - 1)^2$$

$$(\lambda I - A)v = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & 2 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x(\lambda - 1) \\ 2z + y(\lambda - 1) \\ z(\lambda + 1) \end{bmatrix}$$

8. State equation $\mapsto$ State equation

$$\lambda = -1 \qquad \begin{bmatrix} -2x \\ 2z - 2y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y = z \Rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = 1 \qquad \begin{bmatrix} 0 \\ 2z \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow z = 0 \Rightarrow v_2 = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \qquad M^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

8. State equation $\mapsto$ State equation

$$A = \begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \rightarrow \det(\lambda I - A) = (\lambda+1)^4 \rightarrow -1 \text{ of order } \; r = 4$$

$$(\lambda I - A)\,v = \begin{bmatrix} \lambda+2 & 0 & 1 & 0 \\ 0 & \lambda+1 & 0 & 0 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda+1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} z + x(\lambda+2) \\ y(\lambda+1) \\ z\lambda - x \\ t(\lambda+1) - y \end{bmatrix}$$

8. State equation $\mapsto$ State equation

$$\lambda = -1 \Rightarrow \begin{bmatrix} z+x \\ 0 \\ -z-x \\ -y \end{bmatrix} = 0 \Rightarrow x = -z, \; y = 0$$

$$\Rightarrow v = \begin{bmatrix} -a \\ 0 \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{w_1}$$

2 Jordan blocks associated with $\lambda = -1 \Rightarrow$ 2 chains of generalized vectors

$$(-I - A)v_2 = -v_1 \Rightarrow \begin{bmatrix} z+x \\ 0 \\ -z-x \\ -y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow z = -x+1, y = 0 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

8. State equation $\mapsto$ State equation

$$(-I - A)\, v_3 = -v_2 \Rightarrow \begin{bmatrix} z + x \\ 0 \\ -z - x \\ -y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad \text{impossible}$$

$$(-I - A)\, w_2 = -w_1 \Rightarrow \begin{bmatrix} z + x \\ 0 \\ -z - x \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = -z, y = 1 \Rightarrow w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

8. State equation $\mapsto$ State equation

We have :

$$(A+I)v_1 = 0$$
$$(A+I)v_2 = v_1 \Rightarrow (A+I)^2 v_2 = (A+I)v_1 = 0$$

$$(A+I)w_1 = 0$$
$$(A+I)w_2 = w_1 \Rightarrow (A+I)^2 w_2 = (A+I)w_1 = 0$$

A first chain is given by

$$v_2, v_1 = (A-\lambda I)v_2 = (A+I)v_2 \text{ avec } (A+I)^2 v_2 = 0$$

A second chain is given by

$$w_2, w_1 = (A-\lambda I)w_2 = (A+I)w_2 \text{ avec } (A+I)^2 w_2 = 0$$

8. State equation $\mapsto$ State equation

$$M = [v_1, v_2, w_1, w_2] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{M^{-1}} \underbrace{\begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M}$$

$$= \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{J}$$

8. State equation $\mapsto$ State equation

$$A = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \text{ det}(\lambda I - A) = (\lambda + 1)^4 \rightarrow -1 \text{ d'ordre } r = 4$$

$$(\lambda I - A)\, v = \begin{bmatrix} \lambda + 2 & 0 & -1 & 0 \\ -1 & \lambda & 1 & 1 \\ 0 & -1 & \lambda + 1 & 1 \\ 0 & 0 & 0 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x\,(\lambda + 2) - z \\ t - x + z + y\lambda \\ t - y + z\,(\lambda + 1) \\ t\,(\lambda + 1) \end{bmatrix}$$

8. State equation $\mapsto$ State equation

$$\lambda = -1 \Rightarrow \begin{bmatrix} x-z \\ t-x+z-y \\ t-y \\ 0 \end{bmatrix} = 0 \Rightarrow x = z, \; t = y$$

$$\Rightarrow v = \begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{w_1}$$

2 blocs de Jordan associés à $\lambda = -1 \Rightarrow$ 2 chaines de vecteurs propres généralisés

$$(-I - A)v_2 = -v_1 \Rightarrow \begin{bmatrix} x-z \\ t-x+z-y \\ t-y \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow z = x+1, \; y = t+1 \Rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

8. State equation $\mapsto$ State equation

$$(-I - A)v_3 = -v_2 \Rightarrow \begin{bmatrix} x - z \\ t - x + z - y \\ t - y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = z + 1,\ y = t \Rightarrow v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We have :

$$\begin{aligned} (A + I)v_1 &= 0 \\ (A + I)v_2 &= v_1 \\ (A + I)v_3 &= v_2 \Rightarrow (A + I)^3 v_3 = (A + I)^2 v_2 = (A + I)v_1 = 0 \end{aligned}$$

A first chain is given by

$$v_3,\ v_2 = (A + I)v_3,\ v_1 = (A + I)^2 v_3,\ \text{avec}(A + I)^3 v_3 = 0$$

A second chain is given by w_1

8. State equation $\mapsto$ State equation

$$M = [v_1, v_2, v_3, w_1] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

CONTROLLER FORM

If A_c, B_c are the matrices associated to the controller state-space model, we have

$$AM = MA_c \text{ and } B = MB_c$$

leading to (Show it)

$$M = [m_1 \quad \cdots \quad m_n]$$

$$\begin{cases} m_n & = & B \\ m_{n-1} & = & (A + a_{n-1} I_n) B \\ m_{n-2} & = & (A^2 + a_{n-1} A + a_{n-2} I_n) B \\ & \cdots & \\ m_1 & = & (A^{n-1} + a_{n-1} A^{n-2} \cdots + a_1 I_n) B \end{cases}$$

8. State equation ↦ State equation

$$
\begin{aligned}
M &= \begin{bmatrix} m_1 & \cdots & m_n \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{Q_c} \underbrace{\begin{bmatrix} a_1 I & a_2 I & \cdots & a_{n-2} I & a_{n-1} I & I \\ a_2 I & \cdots & a_{n-2} I & a_{n-1} I & I & 0 \\ \vdots & & & & \vdots & \vdots \\ a_{n-2} I & a_{n-1} I & I & \cdots & 0 & 0 \\ a_{n-1} I & I & \cdots & \cdots & 0 & 0 \\ I & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}}_{T}
\end{aligned}
$$

We have

$$\det M = \det Q_c \ \det T$$

and M is invertible if and only if $\det Q_c \neq 0$, then if and only if the system is *controllable*. This structural concept will be defined later

OBSERVER FORM

If C_o, A_o are the matrices associated to the observer state-space model, we have

$$AM = MA_o^T \text{ and } CM = C_o$$

also written as (Show it)

$$A^T(M^T)^{-1} = (M^T)^{-1}A_o \text{ and } C^T = (M^T)^{-1}C_o^T$$

leading to

$$M = \left([m_1 \quad \cdots \quad m_n]^T\right)^{-1}$$

$$\begin{cases} m_1 &= C^T \\ m_2 &= (A^T + a_{n-1}\, I_n)\, C^T \\ m_3 &= ((A^T)^2 + a_{n-1}\, A^T + a_{n-2}\, I_n)\, C^T \\ & \cdots \\ m_n &= ((A^T)^{n-1} + a_{n-1}\, (A^T)^{n-2} \cdots + a_1\, I_n)\, C^T \end{cases}$$

8. State equation $\mapsto$ State equation

$$(M^{-1})^T = \begin{bmatrix} m_1 & \cdots & m_n \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} C^T & A^TC^T & \cdots & (A^T)^{n-1}C^T \end{bmatrix}}_{Q_o^T} \underbrace{\begin{bmatrix} I & a_{n-1}I & a_{n-2}I & \cdots & a_2I & a_1I \\ 0 & I & a_{n-1}I & a_{n-2}I & \cdots & a_2I \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & I & a_{n-1}I & a_{n-2}I \\ 0 & 0 & \cdots & \cdots & I & a_{n-1}I \\ 0 & 0 & \cdots & \cdots & 0 & I \end{bmatrix}}_{T}$$

We have

$$\det(M^{-1})^T = \det {Q_o}^T \ \det T$$

and M is invertible if and only if $\det Q_o \neq 0$, then if and only if the system is *observable*. This structural concept will be defined later

DIAGONAL MODEL FROM CONTROLLER MODEL

- If all the eigenvalues are distinct, it is possible to obtain a diagonal form from the controller
- the matrix M is a **Vandermonde** matrix given by (Show it):

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_1 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda_n^{n-1} \end{bmatrix}$$